Analysis of Two-variable Recurrence Relations with Application to Parameterized Approximations

Ariel Kulik, Technion

Joint work with Hadas Shachnai

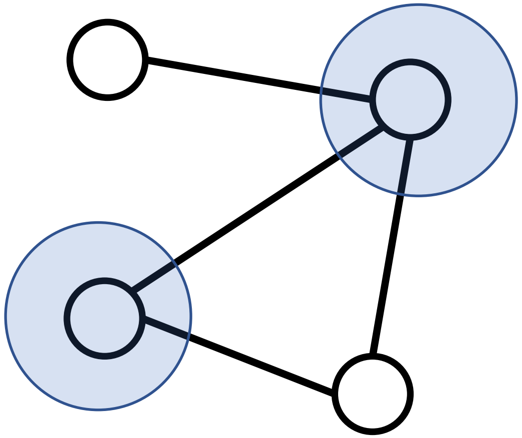
Technion Theory Seminar, Nov 2020

Talk Outline

- Introduction (long)
- Our Results
- Recurrence Relations
- A Faster Algorithm
- Summary and Discussion

Introduction

Vertex Cover (VC)



- An undirected graph $G = (V, E)$
  $S \subseteq V$ is a cover if: $\forall (u, v) \in E$ ➔ $S \cap \{u, v\} \neq \emptyset$
- Decision problem: k -Vertex Cover
  Decide if G has a cover S with $|\mathrm{S}| \leq \mathrm{k}$
- Optimization:
  find a cover S of G such that $|S|$ is minimal
- NP-hard

Approximation

- Find a non-optimal solution in polynomial time.

- A simple polynomial time 2-approximation
  - If G has a cover of size k
  - Finds a vertex cover S with $|S| \leq 2k$

- No $(2 - \varepsilon)$-approximation under UGC

Parameterization

- Associate an instance with a *parameter* k
- Language $L \subseteq \Sigma^*$
- Paramterization $k = \kappa(I)$ for all $I \in \Sigma^*$

- Parameterized Algorithm:
  - Decide if $I \in L$
  - In time $O\big(f(k) \cdot poly(n)\big) = O^*\big(f(k)\big)$
  - $|I| = n,\ k = \kappa(I)$

- Special class of non-polynomial algorithms

Parameterized Vertex Cover

- k -Vertex Cover:
  - Input G and k
  - Decide if G has a cover S with $|\mathrm{S}| \leq \mathrm{k}$

- Standard parameterization $\kappa(G, k) = k$

- A paramaterized $O^*(1.273^k)$ algorithm

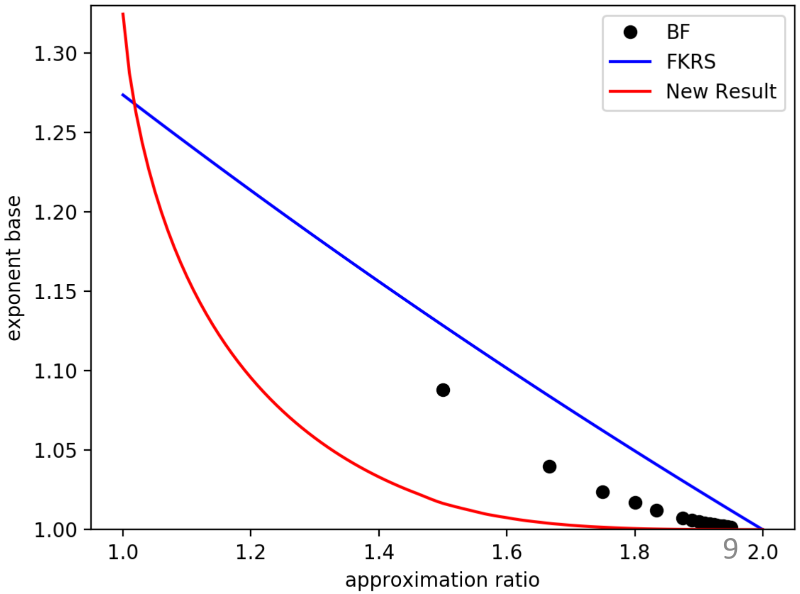
- No $2^{o(k)}$ algorithm under ETH

Parameterized Approximation

- $\alpha \in [1,2]$.
- An algorithm is a parameterized α-approximation for Vertex Cover:
  - If G has a vertex cover of size k
    ➔Returns a vertex cover of size αk
  - Running time $O\left(f(k) \cdot \text{poly}(n)\right)$
- Goal: Tradeoff
  Increase α ➔ Reduce the running time

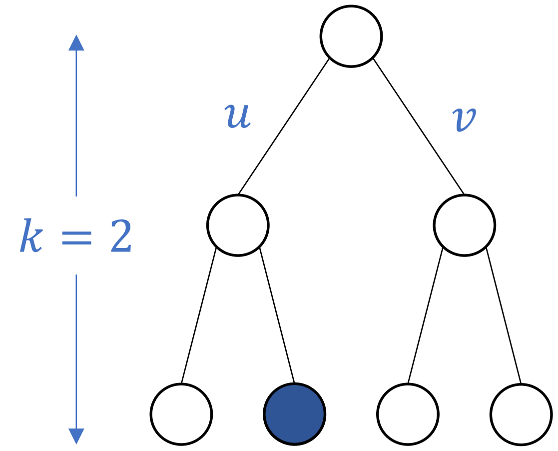
Previous Results

- A parameterized α-approximation with running time $O^*\left((1.2378^{2-\alpha})^k \right)$ [Fellows, K, Rosamond, Shachnai] [Bourgeois, Escoffier, Paschos]
- A parameterized 1.5-approximation in $O^*\left(1.0883^k\right)$ [Brankovic, Fernau]



A simple Algorithm for VC

```
VC2(G, k)
1. If k < 0 return FALSE
2. If E = ∅ return TRUE
3. Pick an edge {u,v}:
   Return VC2(G \ u, k − 1)
     or VC2(G \ v, k − 1)
```

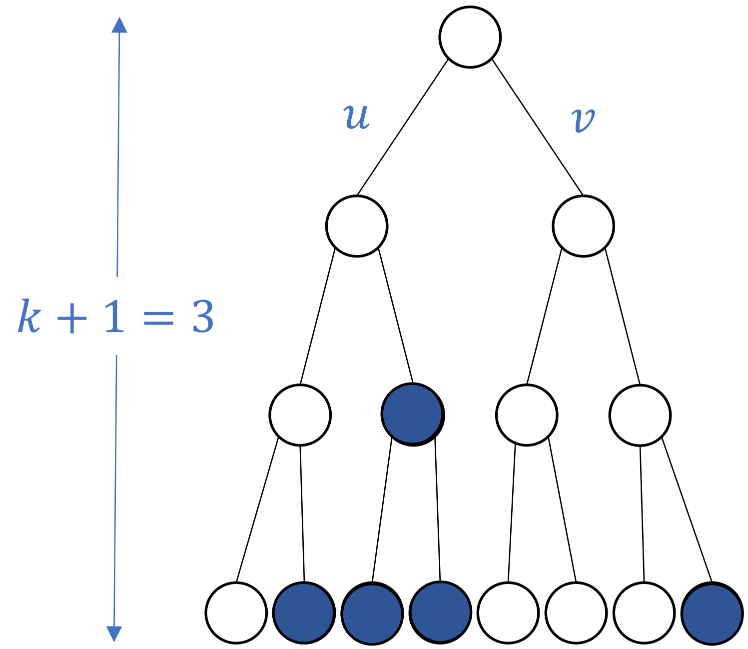


Branching

- An $O^*(2^k)$ algorithm

Approximate Solutions

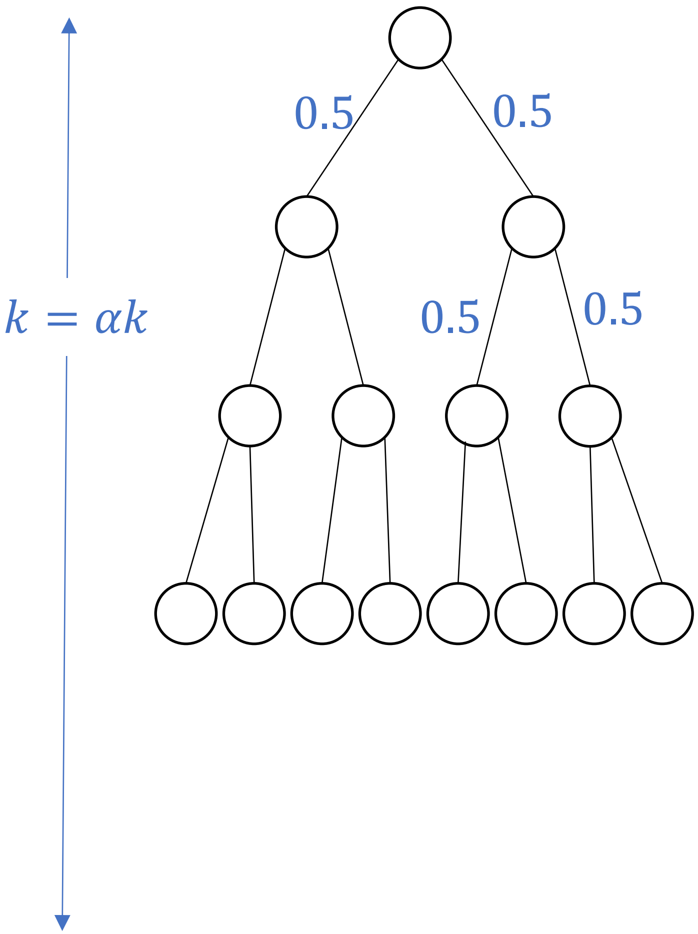
- Pick an edge $\{u, v\}$
- Branch over:
    - u is in the cover
    - v is in the cover
- Repeat $k+1$ times



- Doubled the number of leaves
- $k+2$ "good" leaves

Randomization

- `Pick an edge` $\{u, v\}$
- `With Prob.` 0.5
  `Add` u `to the cover`
  `Else`
  `Add` v `to the cover`
- `Repeat` $\alpha \cdot k$ `times`



- What is the probability a cover is found?

Analysis

```
Pick an edge {u, v}
With Prob. 0.5
    Add u to the cover
Else
    Add v to the cover
Repeat α · k times
```

- Let X_n be a random indicator:
  - $X_n = 1$ the n-th selected vertex reduced the minimal cover size by 1 (or, a cover was already found)
  - Otherwise $X_n = 0$
- $\Pr(X_n = 1 \mid \mathrm{X}_1, \dots, \mathrm{X}_{\mathrm{n}-1}) \geq \frac{1}{2}$

$$\Pr(\text{Found a cover}) \geq \Pr(X_1 + \cdots + X_{\alpha k} \geq k)$$

$$\geq \Pr\left(\text{Binomial}\left(\alpha k, \frac{1}{2}\right) \geq k\right)$$

Analysis- cont'd

$$\Pr(\text{Found a cover}) \geq \Pr\left(\text{Binomial }\left(\alpha k, \frac{1}{2}\right) \geq k\right)$$

$$\geq \frac{1}{(\alpha k + 1)^2} \exp\left(\alpha D\left(\frac{1}{\alpha} \parallel \frac{1}{2}\right)\right)^{-k}$$

$$= \frac{1}{(\alpha k + 1)^2} (c_\alpha)^{-k}$$

D is the Kullback-Leibler Divergen

$$D(a \parallel b) = a \log\left(\frac{a}{b}\right) + (1 -$$

Standard tail bound.
Commonly dervied
using
the Method of Types.

Parameterized Approximation

Run the algorithm:

```
Pick an edge {u, v}
With Prob. 0.5
    Add u to the cover
Else
    Add v to the cover
Repeat α · k times
```

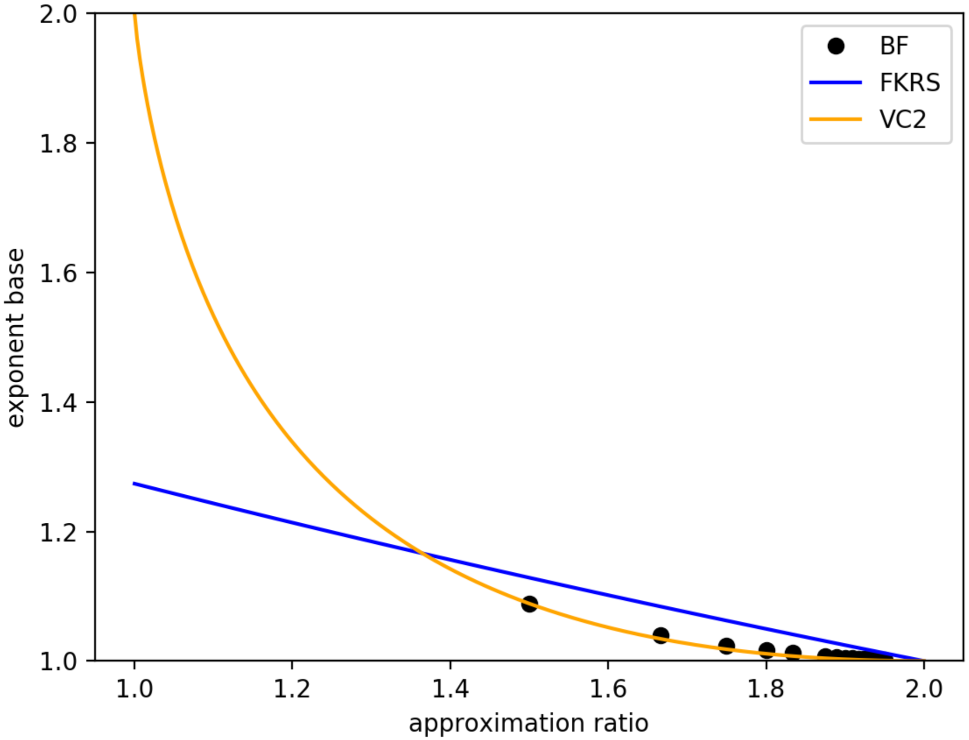
$(\alpha k + 1)^2 (c_\alpha)^k$ times

The graph has a cover of size k

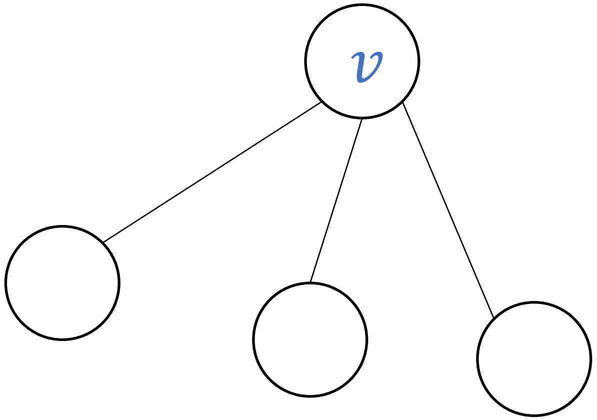
➔With constant probability the algorithm finds a cover of size αk

Running Times

- For $\alpha = 1.5$ the running time is $O^*(1.0887^k)$
- Nearly matches the current best result



Faster Algorithm



- An $O^*(1.46^k)$ algorithm for VC:

If there is $v \in V,\ \deg(v) \geq 3$ branch over:

- v is in the cover
- $N(v)$ is in the cover

Otherwise, $\max \deg(v) \leq 2$, find a minimum vertex cover

Running time: $p(k) = p(k-1) + p(k-3)$

Faster Randomized Algorithm

Fix $\gamma \in (0,1)$.

$$VC3_{\gamma}(G)$$

```
If there is v ∈ V, deg(v) ≥ 3:
   With probability γ
      Add v to the cover
   With probability 1 − γ
      Select S ⊆ N(v), |S|=3
      Add S to the cover
Otherwise, max deg(v) ≤ 2, find a minimum
vertex cover
```

Success Probability

If there is $v \in V$, $\deg(v) \geq 3$:
    With probability γ:
        Add v to the cover.
    With probability $1-\gamma$:
        Select $S \subseteq N(v)$, $|S|=3$
        Add S to the cover.

What is the probabiliy the algorithm returns a cover of size αk?

- $\mathrm{P}(b,k)$- the minimal probabiliy:
  - The algorithm returns a cover of size b - budget
  - Given a graph with cover of size k - parameter

$$p(b,k) = \min\begin{cases} \gamma \cdot p(b-1,k-1) + (1-\gamma) \cdot p(b-3,k) \\ \gamma \cdot p(b-1,k) + (1-\gamma) \cdot p(b-3,k-3) \end{cases}$$

$p(b,k) = 0$ for $b < 0$ and $p(b,0) = 1$ for $b \geq 0$.

Then $\mathrm{P}(b,k) \geq p(b,k)$

Success Probability- cont'd

$$p(b,k) = \min \begin{cases} \gamma \cdot p(b-1,k-1) + (1-\gamma) \cdot p(b-3,k) \\ \gamma \cdot p(b-1,k) + (1-\gamma) \cdot p(b-3,k-3) \end{cases}$$

$p(b,k) = 0$ for $\mathrm{b} < 0$ and $p(b,0) = 1$ for $b \geq 0$.

- $p(b,k)$ can be computed using dynamic progrmming

  ➔ Run the algorithm $\frac{1}{p(\alpha k,k)}$ times

  ➔ α-approximation

- We want to find c such that $p(\alpha k,k) \approx c^{-k}$

$$\lim_{k\to\infty} \frac{1}{k} \log\big(p(\alpha k,k)\big) = ?$$

Our Results

Our results- Highlights

- A solution for a wide familiy of recurrence relations, generalizing:

$$p(b,k) = \min\begin{cases}\gamma \cdot p(b-1,k-1) + (1-\gamma)\cdot p(b-3,k)\\ \gamma \cdot p(b-1,k) + (1-\gamma)\cdot p(b-3,k-3)\end{cases}$$

$p(b,k) = 0$ for $\mathrm{b} < 0$ and $p(b,0) = 1$ for $b \geq 0$.

- Parameterized approximation algorithms for:
  - Vertex Cover
  - 3-Hitting Set

  Significant improvement of the running times.

Generalizing The Reccurence

$$p(b,k) = \min \begin{cases} \gamma \cdot p(b-1,k-1) + (1-\gamma) \cdot p(b-3,k) \\ \gamma \cdot p(b-1,k) + (1-\gamma) \cdot p(b-3,k-3) \end{cases}$$

$$= \min \begin{cases} \bar{\gamma}_1^1 \cdot p(b-\bar{b}_1^1, k-\bar{k}_1^1) + \bar{\gamma}_2^1 \cdot p(b-\bar{b}_2^1, k-\bar{k}_2^1) \\ \bar{\gamma}_1^2 \cdot p(b-\bar{b}_1^2, k-\bar{k}_1^2) + \bar{\gamma}_2^2 \cdot p(b-\bar{b}_2^2, k-\bar{k}_2^2) \end{cases}$$

$$= \min \left\{ \sum_{i=1}^{r_1} \bar{\gamma}_i^1 \cdot p(b-\bar{b}_i^1, k-\bar{k}_i^1) \ , \sum_{i=1}^{r_2} \bar{\gamma}_i^2 \cdot p(b-\bar{b}_i^2, k-\bar{k}_i^2) \right\}$$

$$= \min_{1 \le j \le N} \sum_{i=1}^{r_j} \bar{\gamma}_i^j \cdot p(b-\bar{b}_i^j, k-\bar{k}_i^j)$$

$$\bar{\gamma}^1 = \bar{\gamma}^2 = (\gamma, 1-\gamma), \bar{b}^1 = \bar{b}^2 = (1,3), \bar{k}^1 = (1,0), \bar{k}^2 = (0,3)$$

$$N = 2, \qquad r_1 = r_2 = 2$$

Recurrence Relations

Consider a function $p: \mathbb{Z} \times \mathbb{N} \rightarrow [0,1]$ satisfying*:

$$p(b,k) = \min_{1 \le j \le N} \sum_{i=1}^{r_j} \bar{\gamma}_i^j \cdot p(b - \bar{b}_i^j, k - \bar{k}_i^j)$$

And $p(b,k) = 0$ for $b < 0$, $p(b,0) = 1$ for $b \geq 0$

Where $N \in \mathbb{N}$, for evey $1 \le j \le N$:

- $r_j \in \mathbb{N}$
- $\bar{\gamma}^j \in \mathbb{R}_{>0}^{r_j}$ and $\sum_{i=1}^{r_j} \bar{\gamma}_i^j = 1$
- $\bar{k}^j \in \mathbb{N}^{r_j}$, $\bar{b}^j \in \mathbb{N}_+^{r_j}$

$(\bar{\gamma}^j, \bar{b}^j, \bar{k}^j)$ Defines the "term" $\sum_{i=1}^{r_j} \bar{\gamma}_i^j \cdot p(b - \bar{b}_i^j, k - \bar{k}_i^j)$

Main Theorem

For $\alpha > 0$, the α-Branching number of $\left(\bar{\gamma}^j, \bar{b}^j, \bar{k}^j\right)$ is

$$M_j = f\left(\alpha, \bar{\gamma}^j, \bar{b}^j, \bar{k}^j\right)$$

f is defined by quasiconvex minimzation problem.

➔ Can be computed.

Then,

$$\lim_{k\to\infty} \frac{1}{k} \log\left(p(\alpha k, k)\right) = -\max_{1\le j\le N} M_j$$

$$p(\alpha k, k) \approx \left(\exp\left(\max_{1\le j\le N} M_j\right)\right)^{-k}$$

Main Theorem - Example

$$p(b,k) = \min \begin{cases} \gamma \cdot p(b-1,k-1) + (1-\gamma) \cdot p(b-3,k) \\ \gamma \cdot p(b-1 \ldots \end{cases}$$

$p(b,k) = 0$ for $b < 0$ and

Can be enhanced using quasiconvex optimization

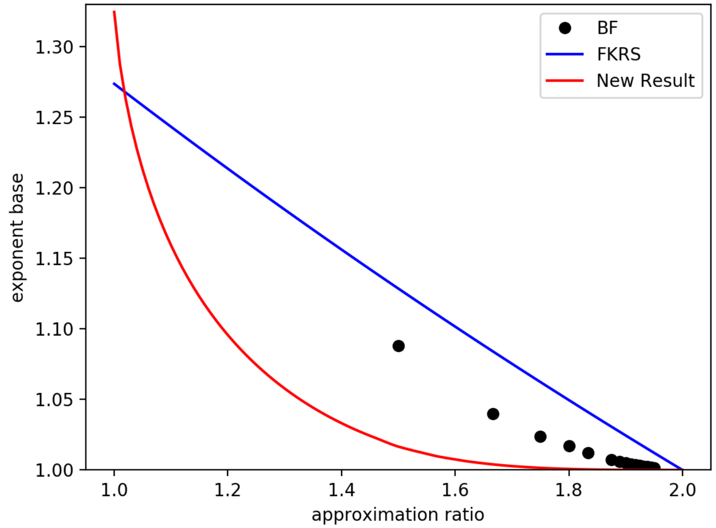
Select $\gamma = 0.7463$. By the theorem,

A 1.5-approximation in time $\frac{1}{p(1.5k,k)} \approx 1.04364^k$

Thus, $p(1.5k,k) \approx \exp(-0.04271)^k \approx 1.04364^{-k}$

$$\bar{\gamma}^1 = \bar{\gamma}^2 = (\gamma, 1-\gamma), \bar{b}^1 = \bar{b}^2 = (1,3), \bar{k}^1 = (1,0), \bar{k}^2 = (0,3)$$

Vertex Cover



Faster parameterized approximation algorithms for vertex cover.

- $\alpha > 1.4$- a variant of $VC3_\gamma$.

  ➔ 1.5-approximation in $O^*(1.017^k)$.

- $\alpha < 1.4$- a variant of a textbook algorithm.

  ➔ 1.1-approximation in $O^*(1.16^k)$.

Additional resuls for 3-Hitting Set

Related Work

Related Work – Recurrences

- Single variable recurrence are studied in introductory courses.

$$p(k) = p(k-1) + p(k-4)$$

- Eppstein, 2006. A different class of multivariate relations.
    - Commonly used in Measure and Conquer.

Recurrences and the Method of Types

Back to Our Algorithm

Fix $\gamma \in (0,1)$.

$VC3_\gamma(G)$

```
If there is v ∈ V, deg(v) ≥ 3:
   With probability γ
      Add v to the cover
   With probability 1 − γ:
      Select S ⊆ N(v), |S|=3
      Add S to the cover
Otherwise, max deg(v) ≤ 2, find a minimum
vertex cover
```

Assumption: v is always in a minimal vertex cover

Random Process

- $P(b, k)$- the probability of finding:
  - A cover of size b or less.
  - Assuming the graph has a cover of size k or less.

- Define $(Y_n)_{n=1}^{\infty}$ by:
  - $Y_n = 1$ if v was selected at the n-th step.
  - Otherwise $Y_n = 2$.

- $\Pr(Y_n = 1) = \gamma, \Pr(Y_n = 2) = 1 - \gamma$

Random Process – Cont'd

- $\bar{k} = (1,0)$.
- $\bar{b} = (1,3)$

$$P(1.5\text{k}, k) = \Pr\left(\exists n: \sum_{\ell=1}^{n} \bar{b}_{Y_\ell} \leq 1.5k \text{ and } \sum_{\ell=1}^{n} \bar{k}_{Y_\ell} \geq k\right)$$

The Method of Types

- $(a_1, \dots, a_n) \in \{1,2\}^n$.
- The Type of $(a_1, \dots, a_n)$ is $\mathbf{T}(\mathrm{a}_1, \dots, a_n) = (T_1, T_2)$,

$$T_i = \frac{|\{\ell |\ a_\ell = i\ \}|}{n}$$ - the frequency of i.

- For example,

$$\mathbf{T}(1,1,1,2,2,1) = \left(\frac{4}{6}, \frac{2}{6}\right)$$

Sanov's Theorem [Sanov, 1961]

$Q \subseteq \{\bar{q} \in \mathbb{R}^2_{\geq 0} \,|\, \sum_{i=1}^{2} \bar{q}_i = 1\}$. $\bar{\gamma} = (\gamma, 1-\gamma)$

$$\Pr(\mathbf{T}(Y_1, \ldots, Y_n) \in Q) \approx \exp(-nc), \quad c = \min_{\bar{q} \in Q} D(\bar{q} \parallel \bar{\gamma})$$

$$\lim_{n \to \infty} \frac{1}{n} \log \Pr(\mathbf{T}(Y_1, \ldots, Y_n) \in Q) = -c$$

D is the Kullback-Leibler Divergence:

$$D(\bar{q} \parallel \bar{\gamma}) = \sum_{i=1}^{2} \bar{q}_i \log \frac{\bar{q}_i}{\bar{\gamma}_i}$$

The point in Q nearest to $\bar{\gamma}$

$$\bar{k} = (1,0), \qquad \bar{b} = (1,3).$$

Using Types

$$P(1.5k, k) = \Pr\left(\exists n: \sum_{\ell=1}^{n} \bar{b}_{Y_\ell} \le 1.5k \text{ and } \sum_{\ell=1}^{n} \bar{k}_{Y_\ell} \ge k\right)$$

$$= \Pr\left(\exists n: T = \mathbf{T}(Y_1, \dots, Y_n), \begin{matrix} n \cdot T_1 \cdot \bar{b}_1 + n \cdot T_2 \cdot \bar{b}_2 \le 1.5k \text{ and} \\ n \cdot T_1 \cdot \bar{k}_1 + n \cdot T_2 \cdot \bar{k}_2 \ge k \end{matrix}\right)$$

$$= \Pr\left(\exists n: T = \mathbf{T}(Y_1, \dots, Y_n), \sum_{i=1}^{2} T_i \cdot \bar{b}_i \le 1.5\frac{k}{n} \text{ and } \sum_{i=1}^{2} T_i \cdot \bar{k}_i \ge \frac{k}{n}\right)$$

$$= \Pr\left(\exists n: \mathbf{T}(Y_1, \dots, Y_n) \in Q_{\frac{k}{n}}\right) \text{ where } Q_\beta = \left\{ \bar{q} \in \mathbb{R}^2_{\ge 0} \;\middle|\; \begin{matrix} \sum_{i=1}^{2} \bar{q}_i = 1 \\ \sum_{i=1}^{2} \bar{q}_i \cdot \bar{b}_i \le 1.5\beta \\ \sum_{i=1}^{2} \bar{q}_i \cdot \bar{k}_i \ge \beta \end{matrix} \right\}$$

Using Types (cont'd)

$$P(1.5k, k) = \Pr\left(\exists n : \mathbf{T}(Y_1, \dots, Y_n) \in Q_{\frac{k}{n}}\right)$$

$$n = \frac{k}{\beta^*}$$

$$\geq \Pr\left(\mathbf{T}\left(Y_1, \dots, Y_{\frac{k}{\beta^*}}\right) \in Q_{\beta^*}\right)$$

$$\approx \exp\left(-\frac{k}{\beta^*} D(\bar{q}^* \parallel \bar{\gamma})\right)$$

matching upper bound.

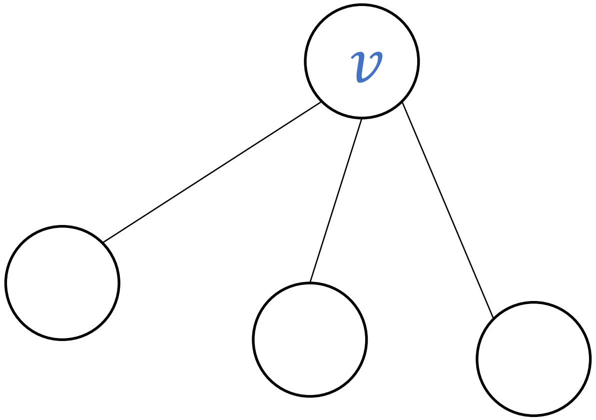
$$\beta^*, \bar{q}^* = \operatorname{argmin}_{\beta, \bar{q} \in Q_\beta} \left(\frac{1}{\beta} D(\bar{q} \parallel \bar{\gamma})\right)$$

The General Solution

- A more complicated random process
  - ➔ Involves all the terms
  - ➔ Cannot apply Sanov's theorem directly
- Properties of types carry over to the new process
- A variant of Sanov's theorem

A Faster Algorithm

Recap

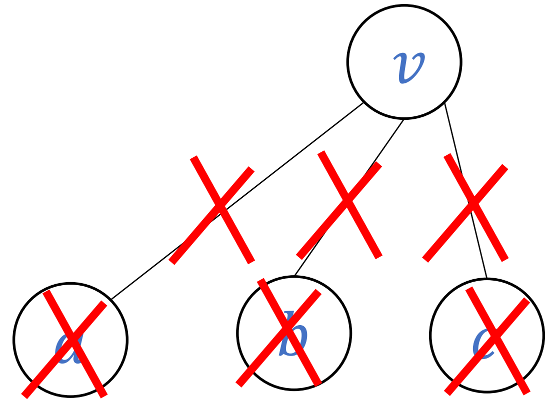


- We showed a 1.5-approximation in $O^*(1.0437^k)$
  - Prob. γ: select v to the cover
  - Prob. $1-\gamma$: select 3 neighbors of v to the cover

$$p(b,k) = \min \begin{cases} \gamma \cdot p(b-1,k-1) + (1-\gamma) \cdot p(b-3,k) \\ \gamma \cdot p(b-1,k) + (1-\gamma) \cdot p(b-3,k-3) \end{cases}$$

$p(b,k) = 0$ for $\mathrm{b} < 0$ and $p(b,0) = 1$ for $b \geq 0$

Refined Analysis



- Focus on:
  - v is in a minimal cover S
  - $N(v) = \{a, b, c\}$

- Two scenarios:
  - v is selected to the cover ➔ k decreases by 1
    $S \setminus \{v\}$ is a cover of $G \setminus \{v\}$
  - $N(v)$ is selected to the cover ➔ k decreases by 1
    $S \setminus \{v\}$ is a cover of $G \setminus N(v)$

An Improved Algorithm

$VC^*(G)$

If there is $v \in V,\ \deg(v) \geq 3$:

- $\deg(v) = 3$
  - $\Pr = \gamma_3$ → Add v
  - $\Pr = 1 - \gamma_3$ → Add $N(v)$
- $\deg(v) \geq 4$:
  - $\Pr = \gamma_4$ → Add v
  - $\Pr = 1 - \gamma_4$ → Add S, $S \subseteq N(v), |S| = 4$

Otherwise, $\max \deg v \leq 2$, find a minimum vertex cover.

Analysis

- probabiliy of finding a cover of size b or less

$$p(b,k) = \min \begin{cases} \gamma_3 \cdot p(b-1,k-1) + (1-\gamma_3) \cdot p(b-3,k-1) \\ \gamma_3 \cdot p(b-1,k) + (1-\gamma_3) \cdot p(b-3,k-3) \\ \gamma_4 \cdot p(b-1,k-1) + (1-\gamma_4) \cdot p(b-4,k) \\ \gamma_4 \cdot p(b-1,k) + \gamma_4 \cdot p(b-4,k-4) \end{cases}$$

$p(b,k) = 0$ for $\mathrm{b} < 0$ and $p(b,0) = 1$ for $b \geq 0$

v is not in an optimal cover, $\deg(v) = 3$.

v is in an optimal cover, $\deg(v) \geq 3$.

Analysis –cont'd

Using the main theorem we can optimize γ_3, γ_4 and get

$$p(1.5k, k) \approx 1.035^{-k}$$

By repeating the algorithm $p(1.5k, k)^{-1}$ times we get:

- 1.5 -approximation with constant probability
- The running time is $O^*(1.035^k)$

➔ Can be generalized

➔ "Incorrect" branching is important

Summary and Discussion

Summary

- Analysis of two variable recurrence relations
  - ➔ Method of Types
  - ➔ Simple formula
  - ➔ In-depth understanding
- Faster parameterized approximation algorithm
  - ➔ Randomized Branching
  - ➔ Significant improvement of running time
  - ➔ Simple algorithms

Thank You