Computing Multilinear Polynomials by Arithmetic Circuits of Bounded Individual Degree

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Technion Theory Lunch
28.10.2020.

[1]Research supported by PBC post doctoral fellowship from Israeli Council of Higher Education.

Strassen's matrix multiplication [Strassen, 1969]

$$\begin{bmatrix} C_{1,1} & C_{1,2} \\ C_{2,1} & C_{2,2} \end{bmatrix} = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} \times \begin{bmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{bmatrix}$$

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$$\begin{aligned}
M_1 &= (A_{1,1} + A_{2,2}) \times (B_{1,1} + B_{2,2}) \\
M_2 &= (A_{2,1} + A_{2,2}) \times B_{1,1} \\
M_3 &= A_{1,1} \times (B_{1,1} - B_{2,2}) \\
M_4 &= A_{2,2} \times (B_{2,1} - B_{1,1}) \\
M_5 &= (A_{1,1} + A_{1,2}) \times B_{1,2} \\
M_6 &= (A_{2,1} - A_{1,2}) \times (B_{1,1} + B_{1,2}) \\
M_7 &= (A_{1,2} - A_{2,2}) \times (B_{2,1} + B_{2,2})
\end{aligned}$$

$$\begin{aligned}
C_{1,1} &= M_1 + M_4 - M_5 + M_7 \\
C_{1,2} &= M_3 + M_5 \\
C_{2,1} &= M_2 + M_4 \\
C_{2,2} &= M_1 - M_2 + M_3 + M_6
\end{aligned}$$

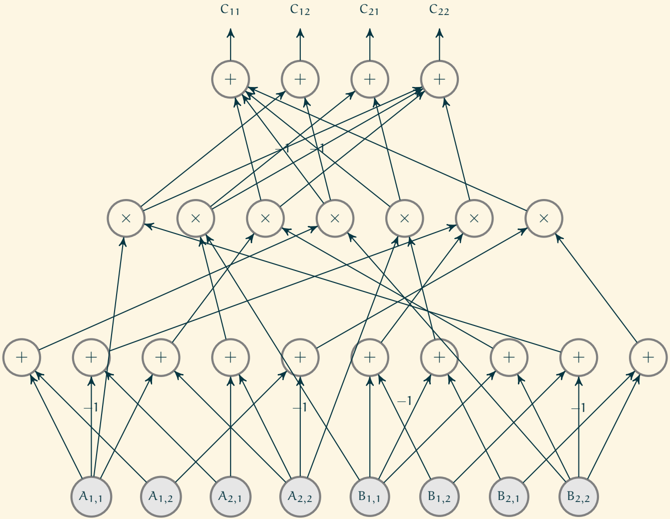


Figure: Strassen's algorithm for multiplication of two 2×2 matrices.

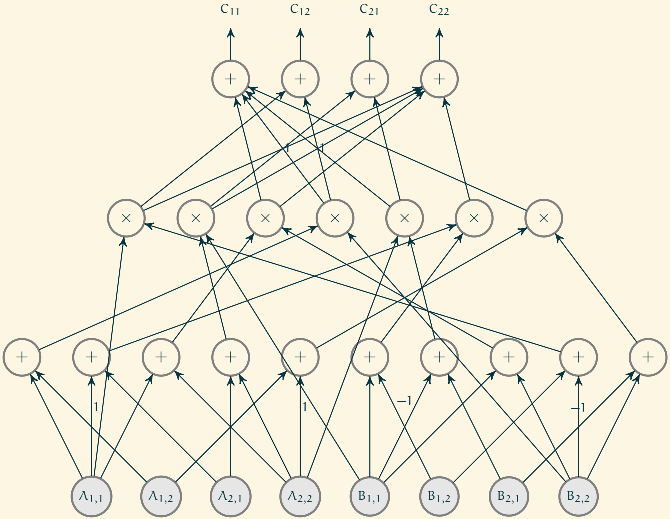


Figure: Strassen's algorithm for multiplication of two 2×2 matrices.

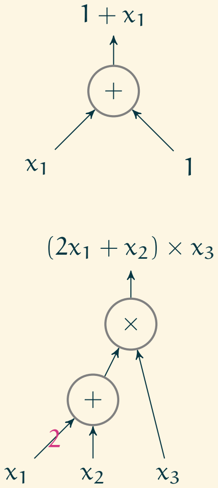
This DAG can be thought of as a "hardwired circuit" for 2×2 matrix multiplication.

Computing polynomials

Definition

An Arithmetic Circuit Φ over the field $\mathbb{F}$ and the set of variables $X = (x_1, x_2, \ldots, x_n)$ is a *directed acyclic graph* as follows:

- Leaf nodes are labelled either by a variable or a field element from $\mathbb{F}$ and the root node outputs the polynomial.
- Every other node is labelled by either $\times$ or $+$.
- The size of Φ is the number of nodes present in it.
- The depth of Φ is the length of the longest leaf to root path.



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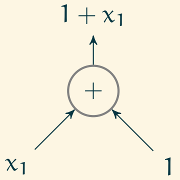

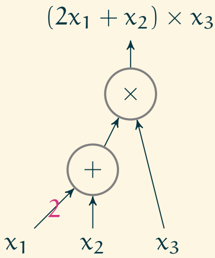

Formulas are circuits whose underlying graph is a tree.

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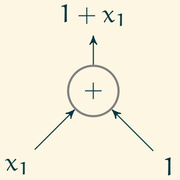

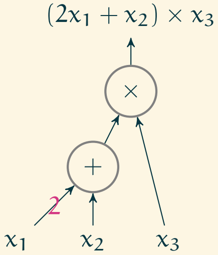

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W.L.O.G we assume arithmetic circuits to be layered: $\Sigma\Pi \cdots \Sigma\Pi$.

Significance of size and depth

- Small circuit size $\implies$ efficient algorithms.

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- Small circuit depth $\implies$ efficient parallel algorithms.

Algebraic P vs Algebraic NP [Valiant, 1979]

Definition (Algebraic P/p-computable/VP)

Class VP consists of all polynomial families $\{f_n\}_{n \geqslant 0}$ of degree $n^{O(1)}$ which can be computed by $n^{O(1)}$ sized arithmetic circuits.

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Class VNP consists of all polynomial families $\{F_n\}_{n \geqslant 0}$ of degree $n^{O(1)}$ which can be expressed as follows.

$$F_n(X) = \sum_{\mathbf{e} \in \{0,1\}^{m(n)}} g_{n,m(n)}(X, \mathbf{e})$$

where $g_{n,m(n)}$ is a polynomial in VP.

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Permanent is a "canonical" polynomial for VNP.

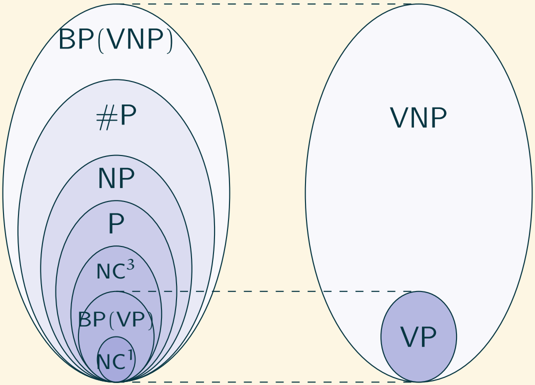
Valiant's hypothesis [Valiant, 1979]

Hypothesis

$$\mathsf{VP} \neq \mathsf{VNP}.$$

That is, Permanent of a generic $n \times n$ matrix cannot be computed by $\mathrm{poly}(n)$-sized arithmetic circuits.

Cook's vs Valiant's hypotheses [Bürgisser, 2000]



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Given a polynomial f, we can assign a corresponding Boolean function $\mathrm{BP}(f)$ to it such that f and $\mathrm{BP}(f)$ agree on evaluations over $\{0,1\}^N$.

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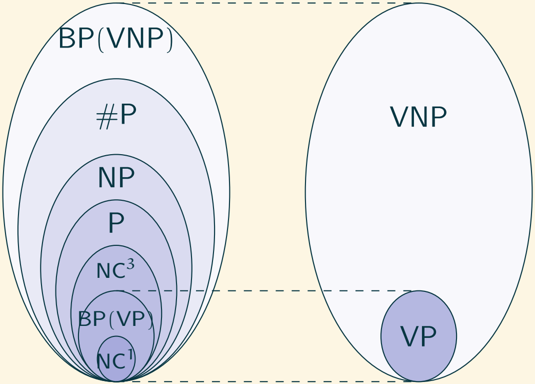

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$\mathsf{VP} \neq \mathsf{VNP}$ can be thought of as a "coarser" separation than $\mathsf{P} \neq \mathsf{NP}$.

Cook's vs Valiant's hypotheses

Theorem [Bürgisser, 2000]

(GRH): If $\mathsf{VP} = \mathsf{VNP}$ then non-uniform $\#\mathsf{P} \subseteq$ non-uniform NC^3.

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- $\mathsf{PH} \subseteq \Sigma_2^p$ [Karp and Lipton, 1980],
- $\mathsf{AM} = \mathsf{MA}$ [Arvind, Köbler, Schöning, and Schuler, 1995].

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Valiant's observations [Valiant, 1992]

- "Since the set of valid algebraic identities in the algebraic model form a proper subset of those in the Boolean setting, lower bound proof for the algebraic setting should be strictly easier."
- "In particular, the main power of the algebraic model derives from the possibility of cancellations."
- Example: Samuelson-Berkowitz method for computing the determinant.

Best known general circuit bounds

- Best known circuit size lower bound is $\Omega(N \log N)$ for a Power Symmetric polynomial [Baur and Strassen, 1983].

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Strategy: Prove lower bounds against restricted models and then extend the understanding to the general setting.

A Restricted Model

Formal degree

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For an arithmetic circuit C and for all $x_i \in X$,

- Formal degree of a leaf node w with respect to x_i,

$$\mathrm{fdeg}_{x_i}(w) = \begin{cases} 1 & \text{if } w \text{ is labelled by variable } x_i, \\ 0 & \text{otherwise.} \end{cases}$$

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- Formal degree of a sum node u with inputs $u_1, \ldots, u_k$, with respect to x_i,

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- Formal degree of a product node v with inputs $v_1, \ldots, v_k$, with respect to x_i,

$$\mathrm{fdeg}_{x_i}(v) = \sum_{j \in [k]} \mathrm{fdeg}_{x_i}(v_j).$$

Multi-r-ic circuits

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When $r = 1$, we have multilinear circuits. We could start proving results for $r = 1$ and then extend these to the setting where $r > 1$.

Lower bounds for syntactically multilinear circuits

- Formulas: $N^{\Omega(\log N)}$ [Raz, 2006; Raz and Yehudayoff, 2008; Dvir, Malod, Perifel, and Yehudayoff, 2012].

- Bounded depth formulas:
  - $2^{\Omega(N^{1/\Delta})}$ [Raz and Yehudayoff, 2009],
  - $2^{\Omega(\Delta N^{1/\Delta})}$ [Chillara, Limaye, and Srinivasan, 2019].

- Circuits:
  - $\Omega\left(N^{1.33}/\log^2 N\right)$ [Raz, Shpilka, and Yehudayoff, 2008],
  - $\Omega\left(N^2/\log^2 N\right)$ [Alon, Kumar, and Volk, 2020].

- Depth four circuits: $N^{\Omega\left(\sqrt{\frac{N}{\log N}}\right)}$ [Raz and Yehudayoff, 2009].

Separations for multilinear circuits

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- **Branching programs vs formulas:** Algebraic branching programs are more powerful than formulas [Dvir, Malod, Perifel, and Yehudayoff, 2012].
- **Separation from general formulas:** Over large fields, general formulas of product-depth $\Delta = o(\log s)$ are more powerful than multilinear formulas of product-depth Δ [Chillara, Limaye, and Srinivasan, 2019].

Hierarchies for multilinear circuits

- **Depth Hierarchy:** Formulas of product-depth Δ are exponentially more powerful than those of product-depth $\Delta - 1$ [Raz and Yehudayoff, 2009; Chillara, Engels, Limaye, and Srinivasan, 2018a].

- **Size Hierarchy:** Formulas of size s are more powerful than the small depth formulas at size $\sqrt{s}$ [Chillara, Limaye, and Srinivasan, 2018b].

Lower bounds for syntactically multi-r-ic circuits

- Homogeneous formulas: $N^{\Omega(\log N)}$ [Kayal, Saha, and Tavenas, 2018].
- Constant Depth Homogeneous Formulas:
  - $2^{\Omega\left(\frac{1}{r}\cdot\left(\frac{N}{2}\right)^{1/\Delta}\right)}$ [Kayal, Saha, and Tavenas, 2018],
  - $2^{\Omega\left(\frac{\Delta}{r}\cdot\left(\frac{Nr}{2}\right)^{1/\Delta}\right)}$ [Chillara, 2019].
- Depth four:
  - Multilinear polynomial: $\left(\frac{n}{r^{1.1}}\right)^{\Omega\left(\sqrt{\frac{d}{r}}\right)}$ where $N = n^2 d$ [Kayal, Saha, and Tavenas, 2018].
  - Multi-r-ic polynomials: For $r = o(N)$,
    - $2^{\Omega(\sqrt{N})}$ [Kayal, Saha, and Tavenas, 2018],
    - $\exp\left(\Omega\left(\sqrt{\frac{N \log N}{r}}\right)\right)$ [Hegde and Saha, 2017].

Depth four multi-r-ic circuits

Definition

A depth four circuit C computes the polynomials of the form

$$f(x_1, \ldots, x_N) = \sum_{i=1}^{s} T_i = \sum_{i=1}^{s} \prod_{j=1}^{d_i} Q_{i,j}(x_1, \ldots, x_N).$$

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For every $T_i = Q_{i,1} \cdot \ldots \cdot Q_{i,D}$ ($i \in [s]$),

- Each variable can appear in at most r many $Q_{i,j}$'s in T_i.
-
$$\forall k \in [N], \sum_{j \in [d_i]} \deg_{x_k}(Q_{i,j}) \leqslant r.$$

A motivation to study depth four circuits

Chasm at depth four

Strong lower bounds against *restricted* depth four circuits imply strong lower bounds against general arithmetic circuits.

- $2^{\omega(\sqrt{d}\log N)}$ against bounded fan-in depth four circuits [Agrawal and Vinay, 2008; Koiran, 2012; Tavenas, 2015],
- $2^{\omega\left(\sqrt{rN\log N}\right)}$ against multi-r-ic depth four circuits [Kumar, de Oliveira, and Saptharishi, 2019].

Previous work for multi-r-ic depth four circuits

Theorem [Kayal, Saha, and Tavenas, 2018]

There exists a fixed constant ν and an explicit multilinear polynomial $Q_{n,d}$ (over $\mathrm{poly}(n, d)$ many variables and degree d) such that for all $d \in \left[\log^2 n, n^{\nu}\right]$ any *syntactically* multi-r-ic depth four circuit computing it must have size $\left(\frac{n}{r^{1.1}}\right)^{\Omega\left(\sqrt{\frac{d}{r}}\right)}$.

We shall first define the explicit polynomial.

Iterated matrix multiplication polynomial

The iterated matrix multiplication polynomial is the $(1,1)$th entry of product of d many generic $n \times n$ matrices $X_1, X_2, \ldots, X_d$ over disjoint set of variables.

$$\mathsf{IMM}_{n,d}(X) = \sum_{i_1, i_2, \ldots, i_{d-1} \in [n]} x^{(1)}_{(1,i_1)} x^{(2)}_{(i_1,i_2)} \ldots x^{(d-1)}_{(i_{(d-2)}, i_{(d-1)})} x^{(d)}_{(i_{(d-1)},1)}$$

where $x^{(k)}_{(i,j)}$ is the variable in X_k indexed by $(i,j) \in [n] \times [n]$.

$$\mathsf{IMM}_{n,d}(X) = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x^{(1)}_{1,1} & \cdots & x^{(1)}_{1,n} \\ \vdots & \ddots & \vdots \\ x^{(1)}_{n,1} & \cdots & x^{(1)}_{n,n} \end{bmatrix} \cdots \begin{bmatrix} x^{(d)}_{1,1} & \cdots & x^{(d)}_{1,n} \\ \vdots & \ddots & \vdots \\ x^{(d)}_{n,1} & \cdots & x^{(d)}_{n,n} \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

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Iterated Matrix Multiplication polynomial $\mathsf{IMM}_{n,d}$ can be expressed in terms of Det_{nd} and thus a lower bound for $\mathsf{IMM}_{n,d}$ implies a lower bound for Det_{nd}.

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- With increasing r, the lower bound deteriorates.
- Lower bound only holds for r that is $o(d)$.

Attempt 1

Theorem [Chillara, 2020a]

There exists a constant $\eta \in (0, 1)$ such that for all $r \leqslant n^{\eta}$, any *syntactically* multi-r-ic depth four circuit computing $\mathsf{IMM}_{n,\Theta(\log^2 n)}$ must have size $n^{\Omega(\log n)}$.

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- Their lower bound deteriorates when r gets closer to $\log^2 n$.
- We give a bound that does not change with increasing r but holds only for degrees that are $\Theta(\log^2 n)$.

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- For the setting of $d = \Theta(\log^2 n)$, [Kayal, Saha, and Tavenas, 2018] gives a lower bound of $n^{\Omega\left(\frac{\log n}{\sqrt{r}}\right)}$ and this is super polynomial only when $r = o(\log^2 n)$.
- Their lower bound deteriorates when r gets closer to $\log^2 n$.
- We give a bound that does not change with increasing r but holds only for degrees that are $\Theta(\log^2 n)$.
- Our bound holds for a value of r that is much larger than d.

Attempt 2

Theorem [Chillara, 2020b]

There exist constants $a \leqslant b \in (0, 1)$ such that for all $d \leqslant n^a$ and $r \leqslant n^b$, any syntactically multi-r-ic depth four circuit computing $\mathsf{IMM}_{n,d}$ must be of size $n^{\Omega(\sqrt{d})}$.

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- Extends [Chillara, 2020a] to give lower bounds that do not deteriorate with increasing values of r, for a wider range of d.

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- Extends [Chillara, 2020a] to give lower bounds that do not deteriorate with increasing values of r, for a wider range of d.
- Though [Kayal, Saha, and Tavenas, 2018] give a lower bound that holds for a "slightly larger" range of d, our lower bound is quantitively better in comparable range of d.

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There exist constants $a \leqslant b \in (0,1)$ such that for all $d \leqslant n^a$ and $r \leqslant n^b$, any syntactically multi-r-ic depth four circuit computing $\mathsf{IMM}_{n,d}$ must be of size $n^{\Omega(\sqrt{d})}$.

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- Though [Kayal, Saha, and Tavenas, 2018] give a lower bound that holds for a "slightly larger" range of d, our lower bound is quantitively better in comparable range of d.
- As with [Chillara, 2020a], we give a bound for a range of $r \geqslant d$.

Shallow separation

Theorem (Implicit in [Chillara, 2020a])

There exists an explicit polynomial Q_n such that

- it can be computed by a depth five multi-r-ic circuit of size $\mathrm{poly}(n)$
- but any depth four multi-r-ic circuit computing it must have size $n^{\Omega(\log n)}$.

Lower bounds for determinant polynomial

Lemma [Valiant, 1979]

$\mathsf{IMM}_{n,d}$ can be expressed as a Determinant of a $nd \times nd$ matrix whose entries are either variables or constants.

Theorem (Implicit in [Kayal, Saha, and Tavenas, 2018; Chillara, 2020b])

There exist fixed constants $a, b \in (0, 1)$ such that for all $r \leqslant N^a$, any syntactically multi-r-ic depth four circuit computing the Determinant of a generic $N \times N$ matrix must have size $2^{\Omega(N^b)}$.

Tools & Techniques

Broad theme of the proofs

Define a suitable complexity measure $\Gamma : \mathbb{F}[X] \mapsto \mathbb{N}$ such that the following holds:

- If f is computed by a small depth four multi-r-ic circuit then $\Gamma(f)$ is *small*.
- For the hard polynomial P, $\Gamma(P)$ is *large*.

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This measure is related to

- Shifted Partial Derivatives measure of [Kayal, 2012],
- Skew Shifted Partial Derivatives measure of [Kayal, Saha, and Tavenas, 2018], and
- Projected Shifted Partial Derivatives measure of [Kayal, Limaye, Saha, and Srinivasan, 2014].

Broad theme of proofs for depth four circuits

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- Show that for all $i \in [s]$, $\Gamma(T_i)$ is not too large, and $\Gamma(f) \gg \Gamma(T_i)$.

Shifted partial derivatives

Dimension of Shifted Partial Derivatives [Kayal, 2012]

For a polynomial $f \in \mathbb{F}[X]$,

$$\partial^{=k} f := \left\{ \partial_m^k f \mid m \text{ is a monomial of degree } k \right\},$$

$$\mathbf{x}^{\leqslant \ell} \cdot \partial^{=k} f := \left\{ m_2 \cdot \partial_{m_1}^k f \mid \deg(m_1) = k \text{ and } \deg(m_2) \leqslant \ell \right\},$$

$$\text{and} \quad \Gamma_{k,\ell}^{[\mathrm{SPD}]}(f) := \dim(\mathbb{F}\text{-span}\left(\mathbf{x}^{\leqslant \ell} \cdot \partial^{=k}(f)\right)).$$

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Theorem [Gupta, Kamath, Kayal, and Saptharishi, 2014]

Let $T = Q_1 \cdot \ldots \cdot Q_D$ where D is "small" and $Q_{i,j}$'s are polynomials of "bounded degree". Then,

- $\Gamma_{k,\ell}^{[\mathrm{SPD}]}(T)$ is not too large for some range of k and ℓ, and
- there exists a polynomial f and parameters k, ℓ such that $\Gamma_{k,\ell}^{[\mathrm{SPD}]}(f) \gg \Gamma_{k,\ell}^{[\mathrm{SPD}]}(T)$.

Multi-r-ic depth four circuits

Let $T = Q_1 \cdot \ldots \cdot Q_D$ be a syntactic multi-r-ic product of polynomials.

Observation 1

Since T is syntactically multi-r-ic , $D \leqslant N \cdot r$.

Multi-r-ic depth four circuits

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Observation 1

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Observation 2

For a random restriction $\rho : X \mapsto \{0, *\}$, with a high probability, $\rho(Q_i)$ is a low degree polynomial. That is, $\rho(T) = Q'_1 \cdot \ldots \cdot Q'_D$ is a product of low degree polynomials.

Multi-r-ic depth four circuits

$$\rho(T) = Q'_1 \cdot \ldots \cdot Q'_D$$

Obstacle: D is still too large

When $D \leqslant N \cdot r$, we get that $\Gamma^{[SPD]}_{k,\ell}(\rho(T)) \gg \Gamma^{[SPD]}_{k,\ell}(\rho(\mathsf{IMM}_{n,d}))$ for all k and ℓ.

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Fix 1: Skew partitions [Kayal, Saha, and Tavenas, 2018]

- Partition X into $Y \sqcup Z$ such that $|Y| \gg |Z|$.
- Under suitable renaming, let

$$\rho(T) = Q_1(Y, Z) \cdot \ldots \cdot Q_t(Y, Z) \cdot R(Y).$$

- Observation: $t \leqslant |Z| \cdot r$.

Shifted skew partial derivatives

Dimension of shifted skew partial derivatives [Kayal, Saha, and Tavenas, 2018]

- Partition X into $Y \sqcup Z$ such that $|Y| \gg |Z|$.
- Under suitable renaming, let

$$\rho(T) = Q_1(Y, Z) \cdot \ldots \cdot Q_t(Y, Z) \cdot R(Y) \quad \text{where} \quad t \leqslant |Z| \cdot r.$$

- $\sigma_Y : \mathbb{F}[Y \sqcup Z] \mapsto \mathbb{F}[Z]$ such that $\sigma_Y(f) \in \mathbb{F}[Z]$. That is, all Y variables are set to 0.

$$\Gamma_{k,\ell}^{[KST]}(f) = \dim \left(\mathbb{F}\text{-span} \left\{ \left(\mathbf{z}^{\leqslant \ell} \cdot \sigma_Y \left(\partial_Y^{=k} f \right) \right) \right\} \right)$$

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Theorem [Kayal, Saha, and Tavenas, 2018]

For a suitable random restriction ρ and carefully chosen values of k and ℓ, $\Gamma^{[KST]}_{k,\ell}(\rho(\mathrm{IMM}_{n,d})) \gg \Gamma^{[KST]}_{k,\ell}(\rho(T))$ and $s \geqslant \left(\frac{n}{r^{1.1}}\right)^{\Omega\left(\sqrt{\frac{d}{r}}\right)}$.

Refined complexity measure [Chillara, 2020a]

Observation

- $\sigma_Y(\partial^{=k}(\rho(\mathsf{IMM}_{n,d})))$ is a multilinear polynomial in $\mathbb{F}[Z]$.
- $\mathbf{z}^{\leqslant \ell} \cdot \sigma_Y\left(\partial_Y^{=k} f\right)$ could potentially lead to non-multilinear polynomials.

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Projected Shifted Skew Partial Derivatives

- Partition X into $Y \sqcup Z$ such that $|Y| \gg |Z|$.
- $\sigma_Y : \mathbb{F}[Y \sqcup Z] \mapsto \mathbb{F}[Z]$ such that $\sigma_Y(f) \in \mathbb{F}[Z]$. That is, all Y variables are set to 0.
- mult : $\mathbb{F}[Z] \mapsto \mathbb{F}[Z]$ sets coefficients of all non-multilinear monomials to 0.

$$\Gamma_{k,\ell}(f) = \dim\left(\mathbb{F}\text{-span}\left\{\mathrm{mult}\left(\mathbf{z}^{\leqslant \ell} \cdot \sigma_Y\left(\partial_Y^{=k} f\right)\right)\right\}\right)$$

Devil lies in the details!

$$\frac{\Gamma_{k,\ell}(\rho(\mathsf{IMM}_{n,d}))}{\Gamma_{k,\ell}(\rho(T_i))} \geqslant n^{\Omega(\sqrt{d})}.$$

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  - [Chillara, 2020b]: Uses a refined and careful counting through Leading Monomial approach (cf. [Kumar and Saraf, 2017]), adapted to this new measure.

Future work

- Prove better bounds against multi-r-ic depth four circuits.
- Combination of syntactic multi-r-ic and homogenity restrictions for formulas computing multi-r-ic polynomials is somewhat like monotone computation (cf. [Jerrum and Snir, 1982; Hrubeš and Yehudayoff, 2011]). Can we
  - weaken the restrictions or
  - prove bounds for multi-$(r-1)$-ic polynomials.
- Polynomial identity testing of depth three and depth four multi-r-ic circuits.

Thank you!