Monotone Submodular Multiple Knapsack

Yaron Fairstein, Ariel Kulik, Joseph (Seffi) Naor,

Danny Raz and Hadas Shachnai

Set function properties

- Let N be a universe of elements and $f: 2^N \to \mathbb{R}_{\geq 0}$ be a set function
- Monotone - if $\forall A \subseteq B$ then $f(A) \leq f(B)$
- Marginal value - for every $A, B \subseteq N$ we define as

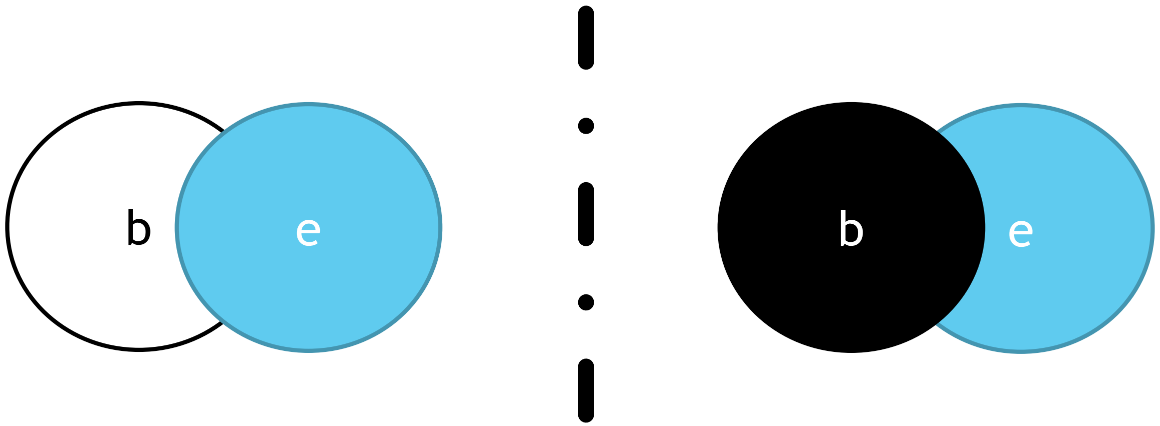
$$f_A(B) = f(A \cup B) - f(A)$$

- Submodular - if for any $A \subseteq B \subseteq N, e \in N \setminus B$

$$f_A(\{e\}) \geq f_B(\{e\})$$

Submodular function examples

- Coverage function (also monotone)
- $A = \emptyset, B = \{b\}$



- Other examples: cut (non monotone), rank (monotone)

Multilinear extension

- The multilinear extension $F: [0,1]^N \to \mathbb{R}_{\geq 0}$ of a f:
  - $F(\vec{x}) = E[f(T)]$, where $T \sim \vec{x}$ ($i \in T$ w.p. x_i)
  - $F(\mathbf{1}_T) = f(T)$
  - Extends to continuous domain
- Continuous greedy can find $\vec{x} \in P$ such that $F(\vec{x}) \geq \left(1 - \frac{1}{e} - \epsilon\right) \cdot OPT$
  - P is the relaxed polytope (describing the constraints)

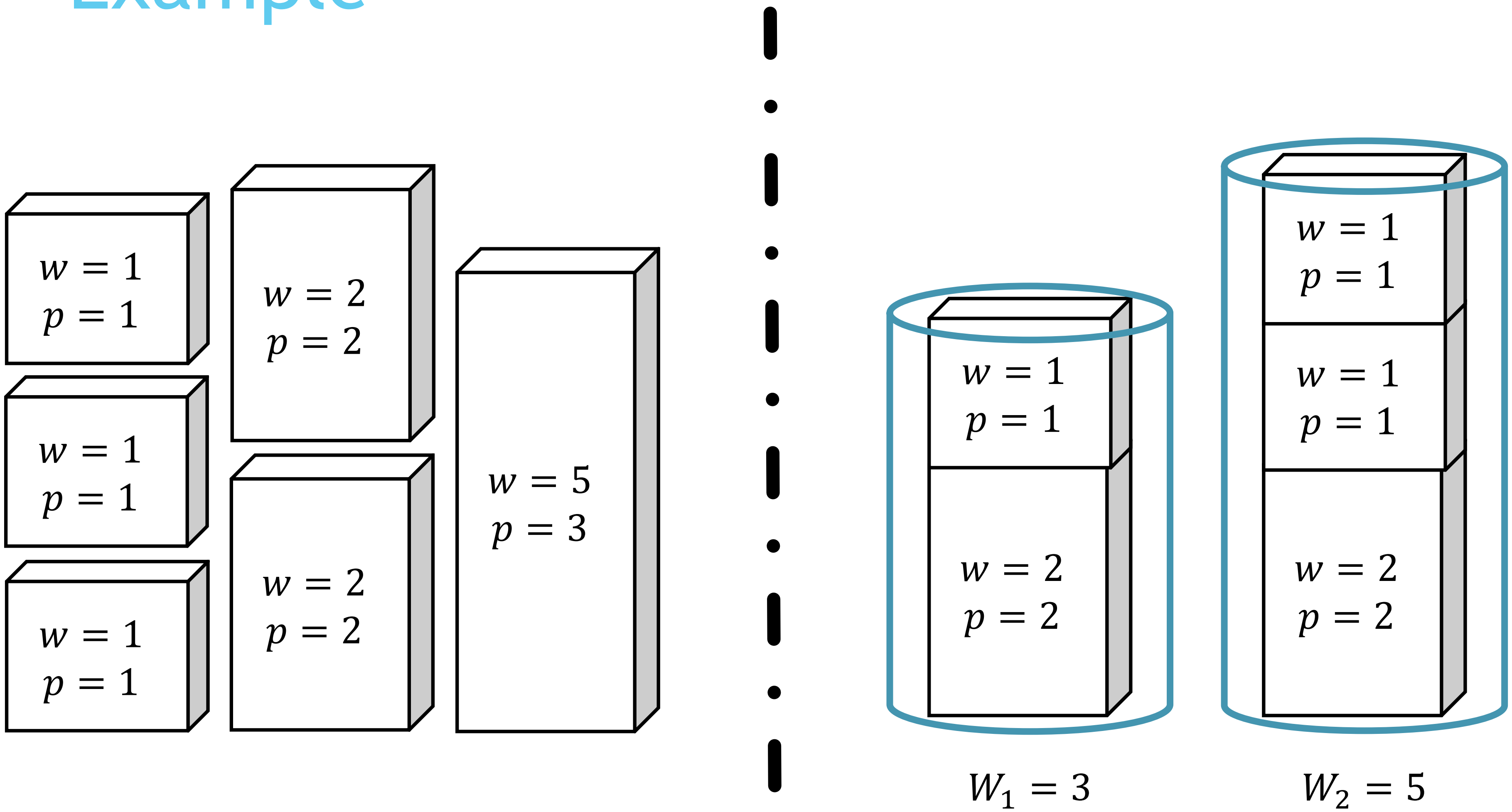
Multiple Knapsack problem (MKP)

- Input:
  - A set of items I with
    - weight w_i
    - profit p_i for each $i \in I$
    - $|I| = n$
  - A set of bins B with
    - capacity W_b for each $b \in B$
    - $|B| = m$

Multiple Knapsack problem (MKP)

- Output:
  - Feasible set $\mathrm{T} \subseteq I$ for which there exists an assignment $A = (A_1, \ldots, A_m)$
    - $\sum_{i \in A_b} w_i \leq W_b$ for all $b \in B$
    - $\bigcup_{b \in B} A_b = T$
- Goal:
  - Find feasible T which maximizes $\sum_{i \in T} p_i$

Example



$w = 1$ $p = 1$
$w = 1$ $p = 1$
$w = 1$ $p = 1$
$w = 2$ $p = 2$
$w = 2$ $p = 2$
$w = 5$ $p = 3$
$w = 1$ $p = 1$
$w = 2$ $p = 2$
$W_1 = 3$
$w = 1$ $p = 1$
$w = 1$ $p = 1$
$w = 2$ $p = 2$
$W_2 = 5$

Monotone Submodular MKP (SMKP)

- Input:
  - MKP constraint:
    - Set of items I with weights w_i
    - Set of bins B with capacities W_b
  - Monotone submodular objective function $f: 2^I \rightarrow \mathbb{R}_{\geq 0}$
- Output:
  - Feasible set $\mathrm{T} \subseteq I$ with assignment A
- Goal:
  - Find feasible set T which maximizes $f(T)$

Our Results

- A random polynomial time $(1-\frac{1}{e}-\epsilon)$-approximation algorithm for Monotone SMKP for any $\epsilon > 0$.

- Known hardness - cannot be approximated within $(1-\frac{1}{e}+\epsilon)$
  - follows hardness subject to cardinality constraint
  - in the oracle model – [Nemhauser, Wolsey. 1978]
  - unless $P \neq NP$ for coverage functions – [Feige. 1998]

Related work

- $\left(1-\frac{1}{e}\right)$-approximation for **constant** number of bins – [Sviridenko. 2003]
- $\left(1-\frac{1}{e}-\epsilon\right)$-approximation for multidimensional knapsack (for constant dimension) – [Kulik, Shachnai, Tamir. 2009]
- Deterministic $\left(1-\frac{1}{e}-\epsilon\right)$-approximation for Monotone SMKP for **uniform bin capacity** – [Sun, Zhang, Zhang. 2020]
  - Randomized $\left(1-\frac{1}{e}-\epsilon\right)$-approximation for **restricted instances** of Monotone SMKP (improved later for general instances)
  - **Parallel work to ours**

Uniform SMKP

- A special case of SMKP: the Uniform SMKP
  - for each pair of bins b_1, b_2 it holds that $W_{b_1} = W_{b_2}$
  - for simplicity assume $W_b = 1$ for all $b \in B$
- For constant $\mu > 0$, split I to sets L, S of large and small items
  - if $w_i \geq \mu$, item i is said to be large
  - else, i is said to be small
- Configuration $c \subseteq L$ is a set of large items s.t. $\sum_{i \in c} w_i \leq 1$
  - $|c| \leq \mu^{-1}$
  - Let C be the set of configurations, then $|\mathrm{C}| \leq n^{\mu^{-1}}$

Relaxation

- New set of items $E = \{e \subseteq I | e \in C \;\; or \;\; e = \{i\} \subseteq S\}$

- At most one “maximal” configuration is assigned to each bin
- Swap all bin constraints by a two dimensional “bin”:
  - The bin (solution) contains at most m configurations
  - The total weight of items and configurations is m

- New objective function $g: 2^E \to \mathbb{R}_{\geq 0}$ defined as $g(T) = f(\bigcup_{e \in T} e)$
  - maintains monotonicity and submodularity

Algorithm

- Phase 1
  - Solve using continuous greedy (w.r.t. multilinear extension), get solution $\vec{x}$
  - Select random set $T \sim \vec{x}$
  - If T violates one of the two constraints, return an empty solution
- Phase 2
  - Assign each configuration $c \in T$ to a different bin
  - Assign small items in T using First-Fit (add bins as necessary)
  - Discard worst bins

Analysis – Phase 1

- Due to guarantees of the continuous greedy and the multilinear extension

$$\mathrm{E}[f(T)] \geq \left(1 - \frac{1}{e} - \epsilon\right) \cdot OPT$$

- What is the probability that T violates a constraint?
- Chernoff bounds yields $\Pr[\text{ T violates a constr}aint\ ] \leq e^{O(-\mu^2 m)}$
  - We lose a factor of $1 - e^{O(-\mu^2 m)}$

Analysis – Phase 2

- What is the loss due to discarded bins/items?

- Once First-Fit finishes bins are almost full
  - the size of items is at most μ
  - free capacity in all but one bin is at most μ
  - assigned weight to added bins is at most $\mu \cdot m$
  - at most $O(\mu^2 \cdot m)$ bins were added
  - $\frac{O(\mu^2 \cdot)}{1+O(\mu^2 \cdot)}$ of the bins are discarded

- Final approximation - $\left(1 - \frac{1}{e} - \epsilon - e^{O(-\mu^2 m)} - O(\mu)\right) \cdot OPT$

Observations

- For some $\epsilon' = \epsilon + e^{O(-\mu^2 m)} + O(\mu)$ we get the desired ratio
- Larger values of m lead to better approximation
  - Small bid assumption
- How well does the algorithm perform on general bin capacities?

SMKP

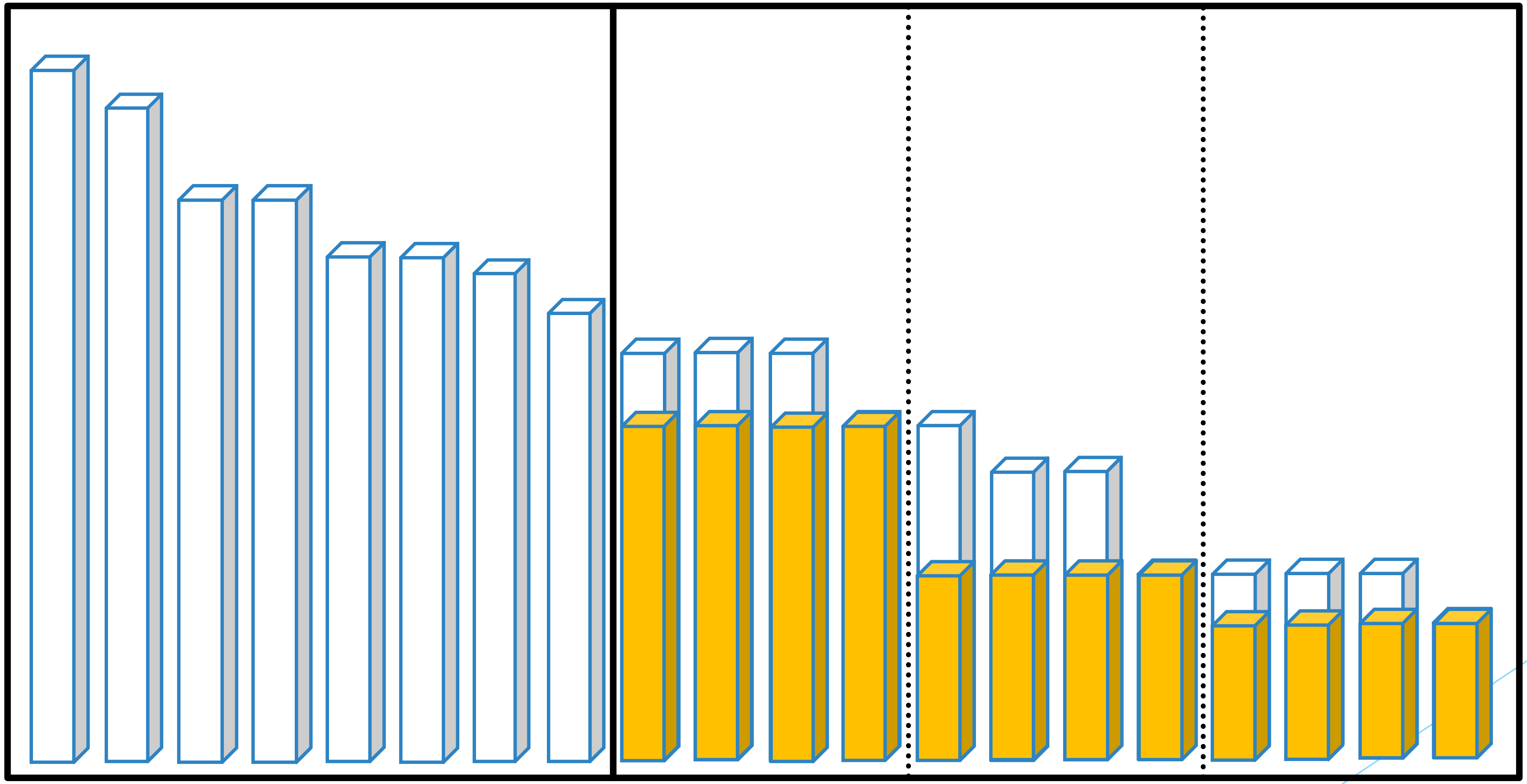
- Block: a set of bins K is called a block if $\forall b_1, b_2 \in K, W_{b_1} = W_{b_2}$
- Let $B = K_1 \cup \cdots \cup K_t$ then the previous algorithm guarantees

$$\left(1 - \frac{1}{e} - \epsilon - \sum_{j=1}^{t} e^{O(-\mu^2 |K_j|)} - O(\mu)\right) \cdot OPT$$

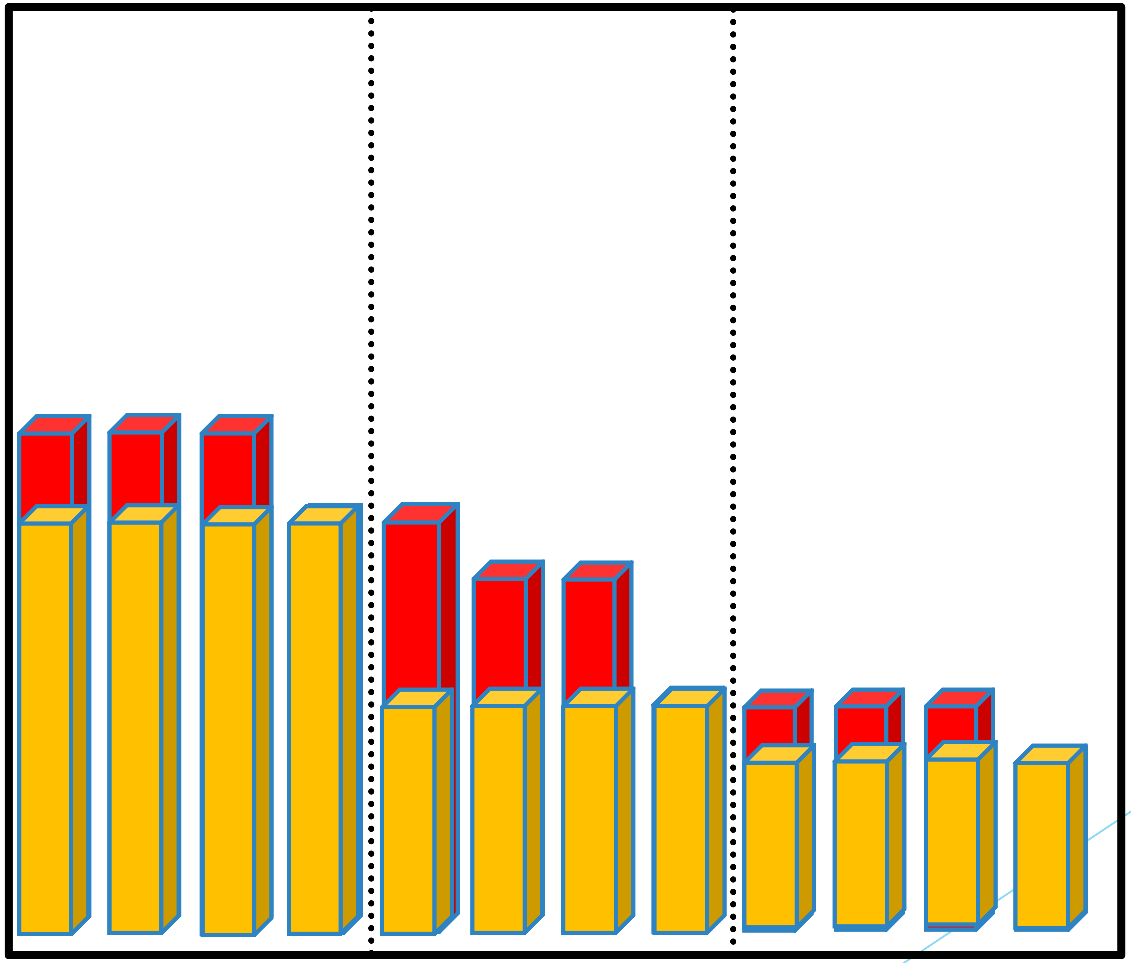
- We might have many small blocks…
  - Note - we can handle few small blocks (enumeration)
- Can we decrease the number of small blocks?

Structuring (by grouping)

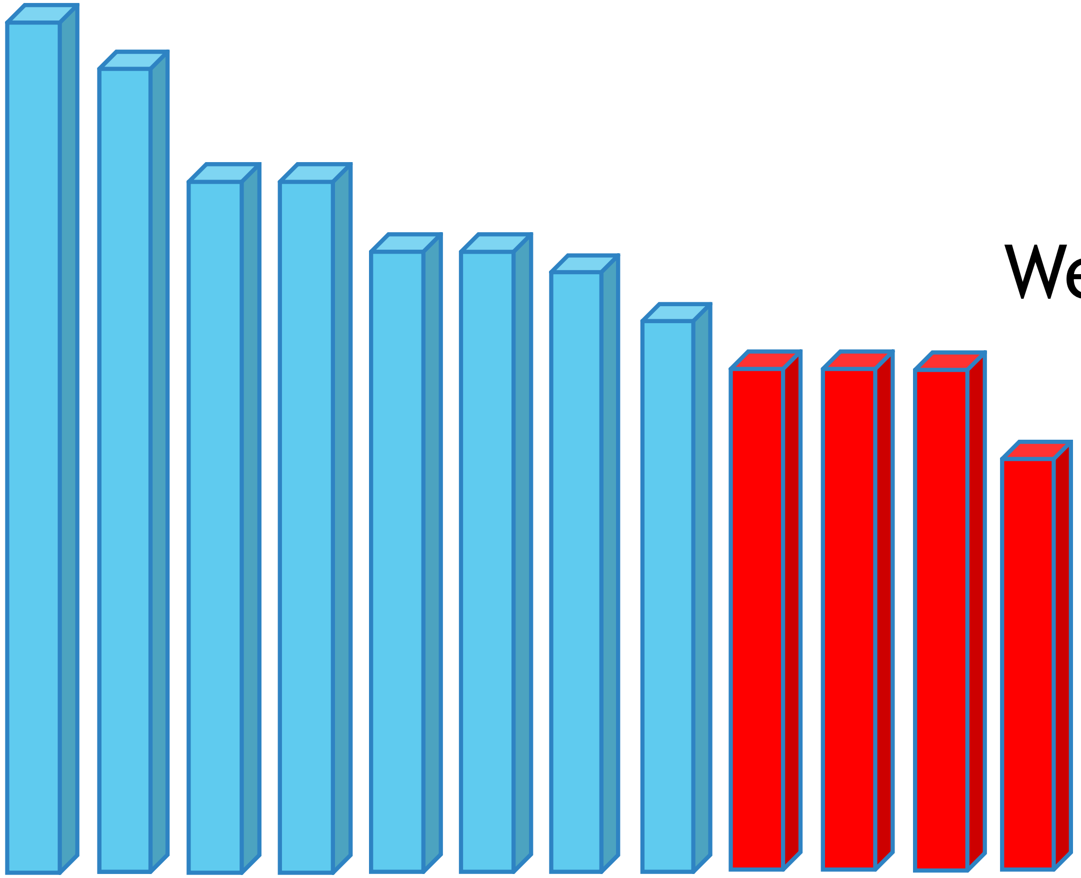
What is the loss?



Structuring loss

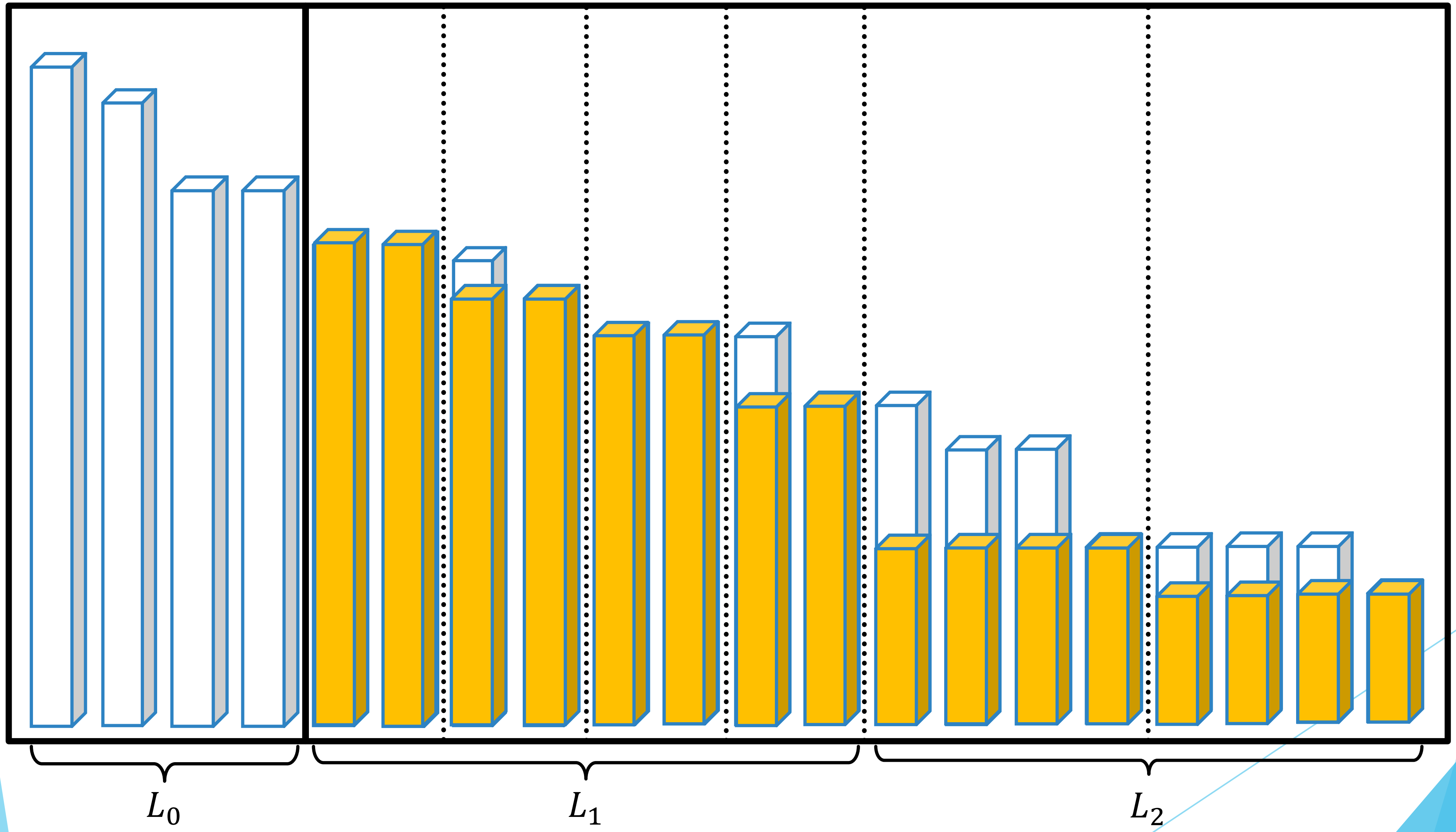


Structuring loss



We lose - $\frac{block\ size}{\#singletons}$

Structuring



N-leveled instances

Level	#Blocks	Block size	#Bins
0	N^2	1	N^2
1	N^2	N	N^3
2	N^2	N^2	N^4
3	N^2	N^3	N^5

- Structuring loss on every level: $1 - \frac{1}{N}$
- Level size grows exponentially
- Small number of small blocks
- Rounding loss converges

Algorithm

- Enumerate – guess the assignment of a constant number of items
- Structure – adjust capacities to get an N-leveled instance
- Solve – using the continuous greedy algorithm (w.r.t. multilinear extension)
- Round – randomly according to the fractional solution
- Assign – to each block using First-Fit

Discussion

- Does the algorithm generalize to natural extensions?
  - Multiple – multiple knapsacks
  - Intersecting matroid constraints
  - Non-monotone objective function
  - Curvature (maximum decrease in marginal value)

- No! because we changed the objective function

Discussion

- $O(1)$ multiple knapsacks constraints
  - Different configurations in different set of knapsacks
- Matroid constraints
  - Matroid properties are lost due to configurations
  - Cardinality constraint – non-uniform “size” for each element
- Non-monotone objective function
  - If not monotone, submodularity isn’t maintained
- Curvature (maximum decrease in marginal value)
  - Curvature of new objective function is always one

Extensions

- **F**, Kulik, Shachnai. "Tight Approximations for Modular and Submodular Optimization with d-Resource Multiple Knapsack Constraints." arXiv.
  - "Insert" the configurations into the constraints
  - Extends to:
    - Multiple – multiple knapsacks constraints
    - Matroid constraints
    - Non-monotone (same approximation as single knapsack constraint)
  - A $\left(1-\frac{1}{e}-o(1)\right)$-approximation for Uniform SMKP

Thank you!