Monotone Submodular Multiple Knapsack

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Set function properties

- Let $N$ be a universe of elements and $f : 2^N \rightarrow \mathbb{R}_{\geq 0}$ be a set function.
- Monotone - if $\forall A \subseteq B$ then $f(A) \leq f(B)$.
- Marginal value - for every $A, B \subseteq N$ we define as
  \[ f_A(B) = f(A \cup B) - f(A) \]
- Submodular - if for any $A \subseteq B \subseteq N, e \in N \setminus B$
  \[ f_A(\{e\}) \geq f_B(\{e\}) \]
Submodular function examples

- Coverage function (also monotone)

- \( A = \emptyset, B = \{b\} \)

- Other examples: cut (non monotone), rank (monotone)
Multilinear extension

- The multilinear extension $F: [0,1]^N \rightarrow \mathbb{R}_{\geq 0}$ of a $f$:
  - $F(\bar{x}) = E[f(T)]$, where $T \sim \bar{x}$ (i \in T \text{ w.p. } x_i)$
  - $F(1_T) = f(T)$
  - Extends to continuous domain

- Continuous greedy can find $\bar{x} \in P$ such that $F(\bar{x}) \geq \left(1 - \frac{1}{e} - \epsilon\right) \cdot OPT$
  - $P$ is the relaxed polytope (describing the constraints)
Multiple Knapsack problem (MKP)

- **Input:**
  - A set of items $I$ with
    - weight $w_i$
    - profit $p_i$ for each $i \in I$
    - $|I| = n$
  - A set of bins $B$ with
    - capacity $W_b$ for each $b \in B$
    - $|B| = m$
Multiple Knapsack problem (MKP)

- Output:
  - Feasible set $T \subseteq I$ for which there exists an assignment $A = (A_1, \ldots, A_m)$
    - $\sum_{i \in A_b} w_i \leq W_b$ for all $b \in B$
    - $\bigcup_{b \in B} A_b = T$

- Goal:
  - Find feasible $T$ which maximizes $\sum_{i \in T} p_i$
Example

\[ w_1 = 1, \quad p_1 = 1 \]
\[ w_2 = 2, \quad p_2 = 2 \]
\[ w_3 = 1, \quad p_3 = 1 \]
\[ w_4 = 2, \quad p_4 = 2 \]
\[ w_5 = 5, \quad p_5 = 3 \]

\[ W_1 = 3 \]
\[ W_2 = 5 \]
Monotone Submodular MKP (SMKP)

- **Input:**
  - MKP constraint:
    - Set of items $I$ with weights $w_i$
    - Set of bins $B$ with capacities $W_b$
  - Monotone submodular objective function $f: 2^I \rightarrow \mathbb{R}_{\geq 0}$

- **Output:**
  - Feasible set $T \subseteq I$ with assignment $A$

- **Goal:**
  - Find feasible set $T$ which maximizes $f(T)$
Our Results

- A random polynomial time \((1 - \frac{1}{e} - \epsilon)\)-approximation algorithm for Monotone SMKP for any \(\epsilon > 0\).

- Known hardness - cannot be approximated within \((1 - \frac{1}{e} + \epsilon)\)
  - follows hardness subject to cardinality constraint
  - in the oracle model - [Nemhauser, Wolsey. 1978]
  - unless \(P \neq NP\) for coverage functions - [Feige. 1998]
Related work

- $(1 - \frac{1}{e})$-approximation for constant number of bins - [Sviridenko. 2003]

- $(1 - \frac{1}{e} - \epsilon)$-approximation for multidimensional knapsack (for constant dimension) - [Kulik, Shachnai, Tamir. 2009]

- Deterministic $(1 - \frac{1}{e} - \epsilon)$-approximation for Monotone SMKP for uniform bin capacity - [Sun, Zhang, Zhang. 2020]
  
  - Randomized $(1 - \frac{1}{e} - \epsilon)$-approximation for restricted instances of Monotone SMKP (improved later for general instances)

- Parallel work to ours
Uniform SMKP

A special case of SMKP: the Uniform SMKP

- for each pair of bins $b_1, b_2$ it holds that $W_{b_1} = W_{b_2}$
- for simplicity assume $W_b = 1$ for all $b \in B$

For constant $\mu > 0$, split $I$ to sets $L, S$ of large and small items

- if $w_i \geq \mu$, item $i$ is said to be large
- else, $i$ is said to be small

Configuration $c \subseteq L$ is a set of large items s.t. $\sum_{i \in c} w_i \leq 1$

- $|c| \leq \mu^{-1}$

Let $C$ be the set of configurations, then $|C| \leq n^{\mu^{-1}}$
Relaxation

- New set of items $E = \{e \subseteq I | e \in C \text{ or } e = \{i\} \subseteq S\}$

- At most one “maximal” configuration is assigned to each bin

- Swap all bin constraints by a two dimensional “bin”:
  - The bin (solution) contains at most $m$ configurations
  - The total weight of items and configurations is $m$

- New objective function $g: 2^E \rightarrow \mathbb{R}_{\geq 0}$ defined as $g(T) = f(\bigcup_{e \in T} e)$
  - maintains monotonicity and submodularity
Algorithm

Phase 1

- Solve using continuous greedy (w.r.t. multilinear extension), get solution $\tilde{x}$
- Select random set $T \sim \tilde{x}$
- If $T$ violates one of the two constraints, return an empty solution

Phase 2

- Assign each configuration $c \in T$ to a different bin
- Assign small items in $T$ using First-Fit (add bins as necessary)
- Discard worst bins
Analysis - Phase 1

- Due to guarantees of the continuous greedy and the multilinear extension

\[ E[f(T)] \geq \left( 1 - \frac{1}{e} - \epsilon \right) \cdot OPT \]

- What is the probability that \( T \) violates a constraint?

- Chernoff bounds yields \( \Pr[ T \text{ violates a constraint } ] \leq e^{O(-\mu^2 m)} \)
  
  - We lose a factor of \( 1 - e^{O(-\mu^2 m)} \)
Analysis - Phase 2

- What is the loss due to discarded bins/items?

- Once First-Fit finishes bins are almost full
  - the size of items is at most $\mu$
  - free capacity in all but one bin is at most $\mu$
  - assigned weight to added bins is at most $\mu \cdot m$
  - at most $O(\mu^2 \cdot m)$ bins were added
  - $\frac{o(\mu^2 \cdot m)}{1+O(\mu^2 \cdot m)}$ of the bins are discarded

- Final approximation - $\left(1 - \frac{1}{e} - \epsilon - e^{O(-\mu^2 m)} - O(\mu)\right) \cdot OPT$
Observations

- For some $\epsilon' = \epsilon + e^{O(-\mu^2 m)} + O(\mu)$ we get the desired ratio

- Larger values of $m$ lead to better approximation
  - Small bid assumption

- How well does the algorithm perform on general bin capacities?
SMKP

- Block: a set of bins $K$ is called a block if $\forall b_1, b_2 \in K, W_{b_1} = W_{b_2}$

- Let $B = K_1 \cup \cdots \cup K_t$ then the previous algorithm guarantees

$$
\left(1 - \frac{1}{e} - \epsilon - \sum_{j=1}^{t} e^{O(-\mu^2 |K_j|)} - O(\mu) \right) \cdot \text{OPT}
$$

- We might have many small blocks...
  - Note - we can handle few small blocks (enumeration)

- Can we decrease the number of small blocks?
Structuring (by grouping)

What is the loss?
Structuring loss
Structuring loss

We lose - $\frac{\text{block size}}{\#\text{singletons}}$
Structuring
N-leveled instances

<table>
<thead>
<tr>
<th>Level</th>
<th>#Blocks</th>
<th>Block size</th>
<th>#Bins</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$N^2$</td>
<td>1</td>
<td>$N^2$</td>
</tr>
<tr>
<td>1</td>
<td>$N^2$</td>
<td>$N$</td>
<td>$N^3$</td>
</tr>
<tr>
<td>2</td>
<td>$N^2$</td>
<td>$N^2$</td>
<td>$N^4$</td>
</tr>
<tr>
<td>3</td>
<td>$N^2$</td>
<td>$N^3$</td>
<td>$N^5$</td>
</tr>
</tbody>
</table>

- Structuring loss on every level: $1 - \frac{1}{N}$
- Level size grows exponentially
- Small number of small blocks
- Rounding loss converges
Algorithm

- **Enumerate** - guess the assignment of a constant number of items

- **Structure** - adjust capacities to get an $N$-leveled instance

- **Solve** - using the continuous greedy algorithm (w.r.t. multilinear extension)

- **Round** - randomly according to the fractional solution

- **Assign** - to each block using First-Fit
Discussion

- Does the algorithm generalize to natural extensions?
  - Multiple - multiple knapsacks
  - Intersecting matroid constraints
  - Non-monotone objective function
  - Curvature (maximum decrease in marginal value)

- No! because we changed the objective function
Discussion

- $O(1)$ multiple knapsacks constraints
  - Different configurations in different set of knapsacks

- Matroid constraints
  - Matroid properties are lost due to configurations
  - Cardinality constraint - non-uniform “size” for each element

- Non-monotone objective function
  - If not monotone, submodularity isn’t maintained

- Curvature (maximum decrease in marginal value)
  - Curvature of new objective function is always one
Extensions


  - “Insert” the configurations into the constraints

  - Extends to:
    - Multiple - multiple knapsacks constraints
    - Matroid constraints
    - Non-monotone (same approximation as single knapsack constraint)

  - $A \left(1 - \frac{1}{e} - o(1)\right)$-approximation for Uniform SMKP
Thank you!