The Metric Relaxation for 0-Extension Admits an $\Omega\left(\log^{2/3} k\right)$ Gap

Nitzan Tur

Joint work with: Roy Schwartz

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- $T = \{t_1, \ldots, t_k\} \subseteq V$ terminals.
- $D : T \times T \to \mathbb{R}^+$ a semi-metric.

Goal: Find $f : V \to T$, identity on T, minimizing:

$$\sum_{(u,v) \in E} w_e \cdot D(f(u), f(v)).$$

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\begin{aligned}
(MET) \qquad \min \quad & \sum_{e=(u,v)\in E} w_e \cdot \delta(u,v) & \\
s.t. \quad & (V,\delta) \text{ is a semi-metric space} & (1)\\
& \delta(t_i,t_j) = D(t_i,t_j) \qquad \forall t_i,t_j \in T, i \neq j & (2)
\end{aligned}
$$

Known Results - Upper Bounds

$O(\log(k))$ [Călinsecu-Karloff-Rabani-05]

$O\left(\frac{\log(k)}{\log\log(k)}\right)$ [Fakcharoenphol-Harrelson-Rao-Talwar-03]

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Above algorithms consist of two steps:

1. Select "scale" for each vertex.
2. Decompose the metric δ in each scale.

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- Admits integrality gap of $\Omega(\sqrt{\log k})$ [Karloff-Khot-Mehta-Rabani-09].

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Question: bridge the gap between $O\left(\frac{\log(k)}{\log\log(k)}\right)$ and $\Omega(\sqrt{\log k})$ for (MET)?

Our Results

Theorem [Schwartz-T-20]

For every k, (MET) admits an integrality gap of $\Omega(\log^{2/3}(k))$ for 0-Extension.

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Proof Overview:

- Construction of graph extensions.
- Small gap implies that graph extensions "split".
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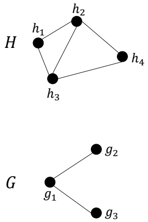

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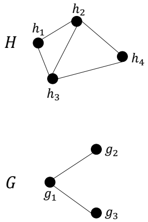

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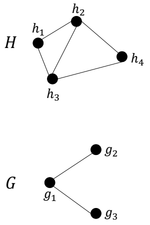

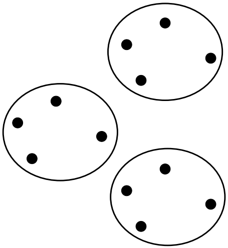

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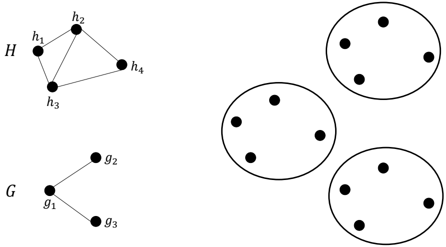

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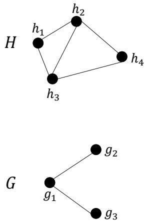

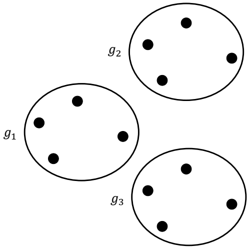

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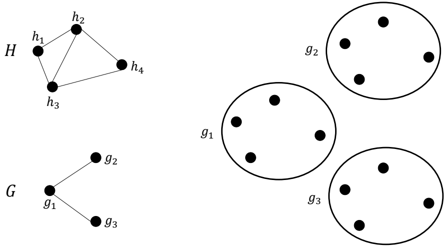

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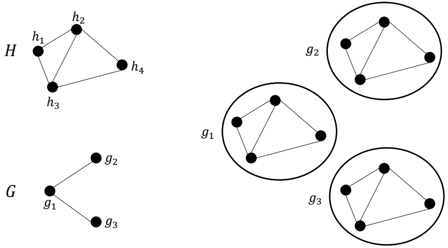

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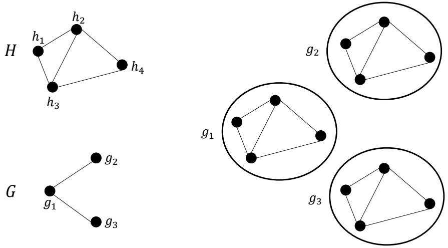

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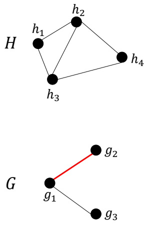

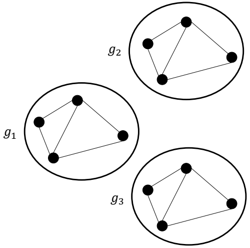

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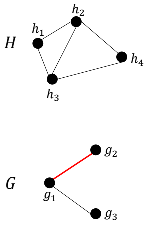

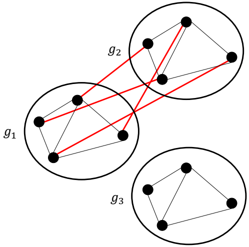

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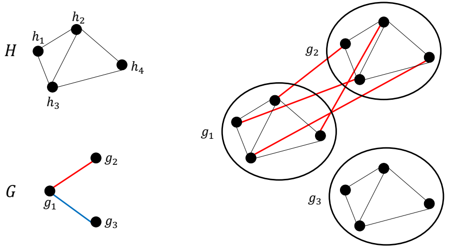

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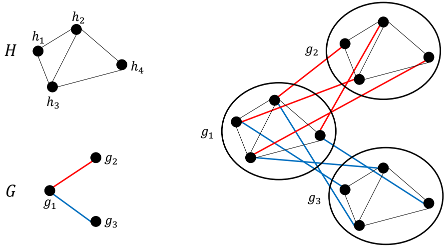

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3. Relates to group extensions:

$$G \text{ and } H \text{ are Cayley graphs and } K \text{ is a group extension of } G \text{ by } H$$
$$\Downarrow$$
$$K\text{'s Cayley graph is in the support of } \mathrm{Ext}(G, H)$$

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Notes:

- Need to quantify **most** and **close**.
- Captures split extensions of groups.

The Instance

Instance Definition

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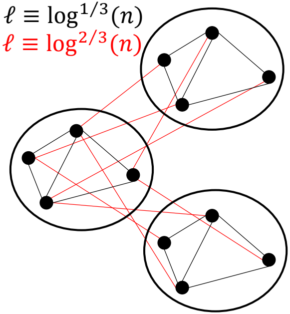

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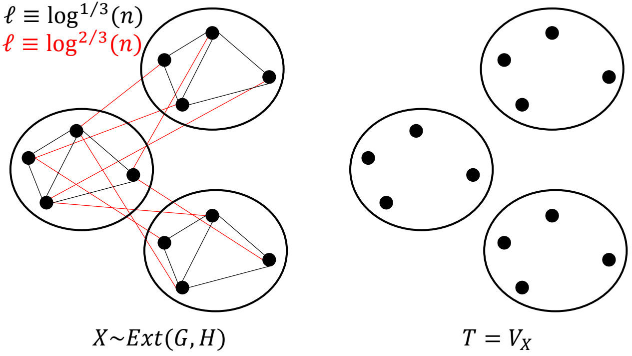

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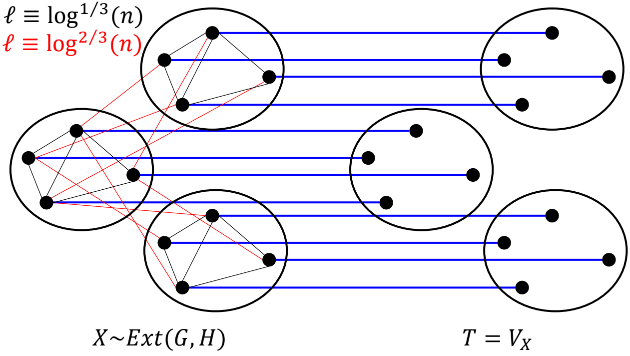

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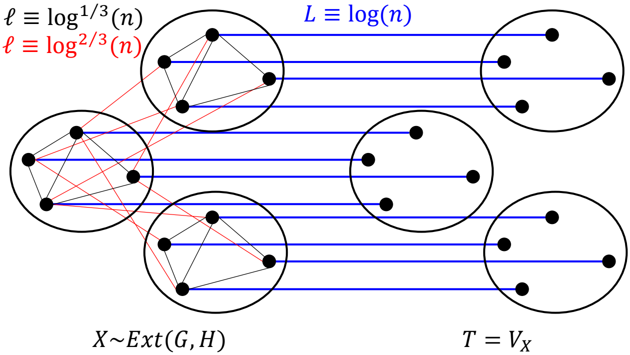

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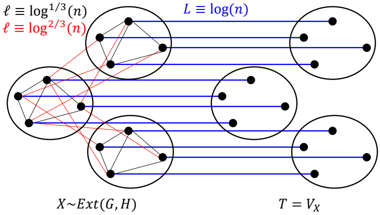

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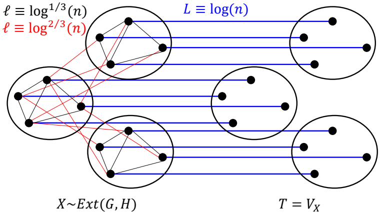

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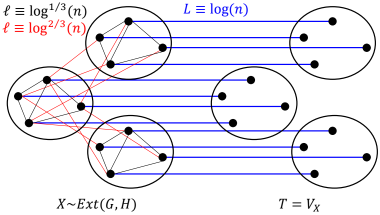

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Conclusion: $\delta(f(u), f(v)) \leq \varepsilon \log^{2/3}(n) \delta(u, v)$ for $1 - \varepsilon$ of the edges.

Split - Existence of Representatives

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Given $X \sim \text{Ext}(G, H)$ a split assigns to **most** clouds g a **representative** $f(g) \in V_X$ where:

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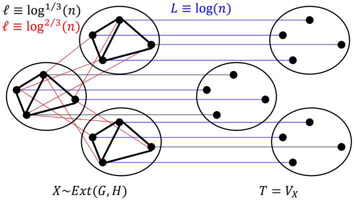

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- Intra-cloud neighbors distance is $\log^{1/3}(n)$.

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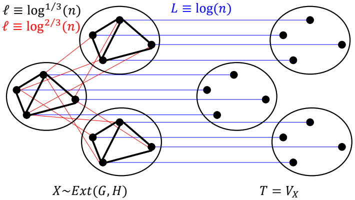

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Split - Existence of Representatives

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Given $X \sim \text{Ext}(G, H)$ a split assigns to **most** clouds g a **representative** $f(g) \in V_X$ where:

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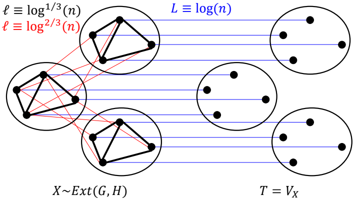

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Most clouds have a consensus and this consensus is the representative.

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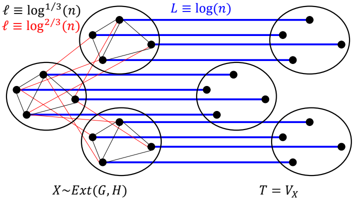

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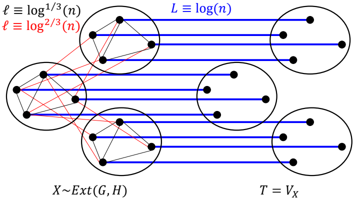

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Cloud of $f(g)$ is $\underbrace{\varepsilon \log^{2/3}(n)}_{\text{gap}} \cdot \log(n) / \log^{2/3}(n) = \varepsilon \log(n)$ **hops** away in G from g.

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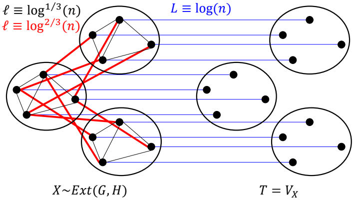

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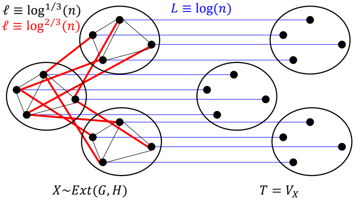

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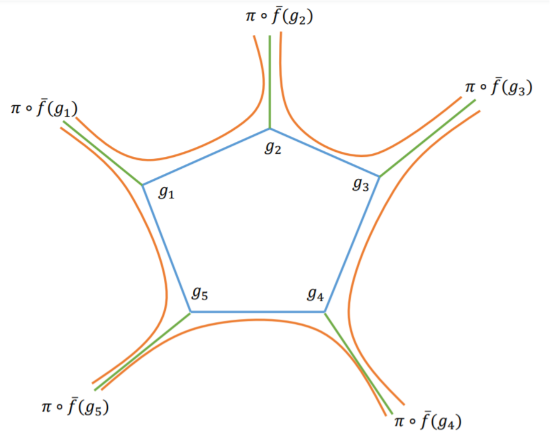
Algebraic topology intuition: $\pi \circ f$ is a homeomorphism $\Rightarrow$ it preserves the first homology.

Cycle-Homeomorphism

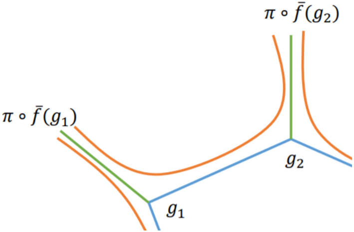


- $\pi : V_X \to V_G$, the natural projection.
- $f : V_G \to V_X$, the representative map.
- f induces a map $\bar{f} : E_G \to \mathbb{F}_2^{E_X}$, "the short path map"

We call f a *Cycle-Homeomorphism* if $\pi \circ \bar{f} : \mathbb{F}_2^{E_G} \to \mathbb{F}_2^{E_G}$ is identity on cycles.

Cycle-Homeomorphism

Let g_1, g_2 neighboring clouds.



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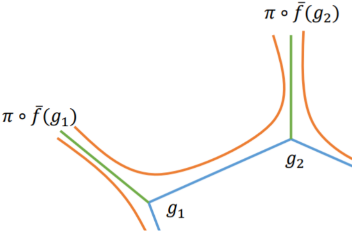

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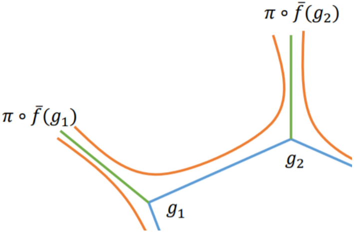

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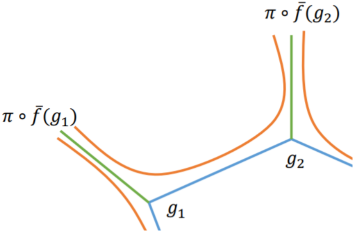

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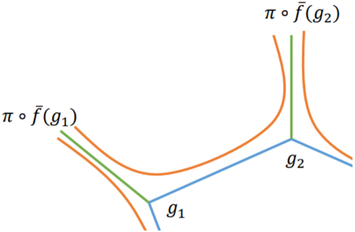

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- The cycle $g_1 \to g_2 \to \pi \circ \bar{f}(g_2) \to \pi \circ \bar{f}(g_1) \to g_1$ has $O(\varepsilon \log(n))$ edges.
- The girth of G has $\Omega(\log(n))$.
- This cycle is trivial.
- f is cycle-homeomorphism.

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- If $|E_G| \geq 2|V_G|$, then split should not exist (via union bound).

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2. The constraints are not probabilistically independent:
   - Define a suitable combinatorial structure that allows enough independence.
   - Linearly independent (modulo 2) cycles imply probabilistic independence.

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A certificate encodes a "formal roadmap" of:

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A vertex of the above graph is an Inner Connected Component.

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- Each closed cycle is correct "independently".

Certificates - Bounding Probability (cont.)

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Certificates - Bounding Probability (cont.)

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We are done by a union bound as:

$$\underbrace{n^{-\chi\cdot(1-O(\varepsilon))}}_{\text{probability}} \cdot \underbrace{n^{(1+O(\varepsilon))}}_{\text{no. certificates}} \leq n^{-(\chi-n)(1-O(\varepsilon))} \ll 1$$

Questions?